Bäcklund transformations: An introduction

Costas J. Papachristou[1]

Hellenic Naval Academy

The concept of a Bäcklund transformation (BT) is introduced. Certain applications of BTs – both older and more recent ones – are discussed.

1. Introduction

Given a difficult problem in mathematics we always look for some way to *transform* it to another problem that is easier to solve. Thus, for example, we seek an integrating factor that might transform a first-order ordinary differential equation into an exact one (or would reduce the order of a higher-order differential equation, in the more general case).

A notoriously difficult problem in the theory of partial differential equations (PDEs) is the case of *nonlinear* PDEs. In contrast to the case of linear PDEs, there is no general method for solving nonlinear ones. Thus, given a nonlinear PDE we look for ways to associate it with some other PDE (preferably a linear one!) whose solutions are already known. For example, the *Burgers equation* $u_t=u_{xx}+2uu_x$ is a nonlinear PDE for the function $u(x,t)$ (subscripts denote partial derivatives with respect to the indicated variables). This PDE can be transformed into the linear *heat equation* $v_t=v_{xx}$ by using the so-called *Cole-Hopf transformation* $u=v_x/v$. As can be shown, if $v(x,t)$ is a solution of the heat equation then $u(x,t)$ is a solution of the Burgers equation (the converse is not true in general).

Bäcklund transformations (BTs) were originally devised mainly as a tool for obtaining solutions of nonlinear PDEs (see [1] and the references therein). They were later also proven useful as *recursion operators* for constructing infinite sequences of nonlocal symmetries and conservation laws of certain types of PDEs [2–6].

In simple terms, a BT is a system of PDEs connecting two fields that are required to independently satisfy two respective PDEs [call them (a) and (b)] in order for the system to be integrable for either field. We say that the PDEs (a) and (b) are *integrability conditions* for self-consistency of the BT. If a solution of PDE (a) is known, then a solution of PDE (b) is obtained simply by integrating the BT, without having to actually solve the latter PDE (which, presumably, would be a harder task). In the case where the two fields satisfy the same PDE, the *auto-BT* produces new solutions of this PDE from old ones.

As described above, a BT is an auxiliary tool for finding solutions of a given (usually nonlinear) PDE, using known solutions of the same or another PDE. Now, suppose the BT itself is the differential system whose solutions we are looking for. As will be seen, one possible way to solve this problem is to first seek parameter-dependent solutions of both integrability conditions of the BT. By properly matching the parameters (provided this is possible) a solution of the given differential system is obtained.

The above method is particularly effective in *linear* problems, given that parametric solutions of linear PDEs are generally easier to find. An important paradigm of a BT associated with a linear problem is offered by the Maxwell system of equations of

[1] cjpapachristou@gmail.com

electromagnetism [7,8]. As is well known, the consistency of this system demands that both the electric and the magnetic field independently satisfy a respective wave equation. The wave equations for the two fields have known, parameter-dependent solutions; namely, monochromatic plane waves with arbitrary amplitudes, frequencies and wave vectors (the "parameters" of the problem). By inserting these solutions into the Maxwell system, one may find the appropriate constraints for the parameters in order for the plane waves to also be solutions of Maxwell's equations.

In Section 2 we review the classical concept of a BT. The solution-generating process by using a BT is demonstrated in a number of examples.

In Sec. 3 a different perception of a BT is presented, according to which it is the BT itself whose solutions are sought. The concept of *parametric conjugate solutions* is introduced.

In Sec. 4 we examine the connection between BTs and recursion operators for generating infinite sequences of nonlocal symmetries of PDEs.

2. Bäcklund transformations and generation of solutions

Let $u(x,t)$ be a function of two variables. For the partial derivatives of u the following notation will be used:

$$\frac{\partial u}{\partial x} = \partial_x u = u_x \ , \ \ \frac{\partial u}{\partial t} = \partial_t u = u_t \ , \ \ \frac{\partial^2 u}{\partial x^2} = u_{xx} \ , \ \ \frac{\partial^2 u}{\partial t^2} = u_{tt} \ , \ \ \frac{\partial^2 u}{\partial x \partial t} = u_{xt} \ ,$$

etc. In general, a subscript will denote partial differentiation with respect to the indicated variable.

Let F be a function of x, t, u, as well as of a number of partial derivatives of u. We will denote this type of dependence by writing

$$F(x,t,u,u_x,u_t,u_{xx},u_{tt},u_{xt},\cdots) \equiv F[u] \ .$$

We also write

$$F_x = \partial_x F = \partial F / \partial x \ , \quad F_t = \partial_t F = \partial F / \partial t \ , \quad F_u = \partial_u F = \partial F / \partial u \ ,$$

etc. Note that in determining F_x and F_t we must take into account both the *explicit* and the *implicit* (through u and its partial derivatives) dependence of F on x and t. As an example, for $F[u] = 3xtu^2$ we have $F_x = 3tu^2 + 6xtuu_x$ and $F_t = 3xu^2 + 6xtuu_t$.

Consider now two partial differential equations (PDEs) $P[u]=0$ and $Q[v]=0$ for the unknown functions u and v, respectively, where the bracket notation introduced above is adopted. Both u and v are functions of two variables x, t. Independently, for the moment, consider also a pair of coupled PDEs for u and v:

$$B_1[u,v]=0 \quad (a) \qquad B_2[u,v]=0 \quad (b) \tag{1}$$

where the expressions $B_i\,[u,v]$ (i=1,2) may contain u, v as well as partial derivatives of u and v with respect to x and t. We note that u appears in both equations (a) and (b). The question then is: if we find an expression for u by integrating (a) for a given v, will it match the corresponding expression for u found by integrating (b) for the same v? The answer is that, in order that (a) and (b) be consistent with each other for solution for u, the function v must be properly chosen so as to satisfy a certain *consistency condition* (or *integrability condition* or *compatibility condition*).

By a similar reasoning, in order that (*a*) and (*b*) in (1) be mutually consistent for solution for *v*, for some given *u*, the function *u* must now itself satisfy a corresponding integrability condition.

If it happens that the two consistency conditions for integrability of the system (1) are precisely the PDEs $P[u]=0$ and $Q[v]=0$, we say that the above system constitutes a *Bäcklund transformation* (BT) connecting solutions of $P[u]=0$ with solutions of $Q[v]=0$. In the special case where $P \equiv Q$, i.e., when *u* and *v* satisfy *the same* PDE, the system (1) is called an *auto-Bäcklund* transformation (auto-BT) for this PDE.

Suppose now that we seek solutions of the PDE $P[u]=0$. Assume that we are able to find a BT connecting solutions *u* of this equation with solutions *v* of the PDE $Q[v]=0$ (if $P \equiv Q$, the auto-BT connects solutions *u* and *v* of the same PDE) and let $v=v_0(x,t)$ be some known solution of $Q[v]=0$. The BT is then a system of PDEs for the unknown *u*,

$$B_i\,[u, v_0] = 0 \;, \quad i = 1, 2 \tag{2}$$

The system (2) is integrable for *u*, given that the function v_0 satisfies *a priori* the required integrability condition $Q[v]=0$. The solution *u* then of the system satisfies the PDE $P[u]=0$. Thus a solution $u(x,t)$ of the latter PDE is found without actually solving the equation itself, simply by integrating the BT (2) with respect to *u*. Of course, this method will be useful provided that integrating the system (2) for *u* is simpler than integrating the PDE $P[u]=0$ itself. If the transformation (2) is an auto-BT for the PDE $P[u]=0$, then, starting with a known solution $v_0(x,t)$ of this equation and integrating the system (2), we find another solution $u(x,t)$ of the same equation.

Let us see some examples of the use of a BT to generate solutions of a PDE:

1. The *Cauchy-Riemann relations* of Complex Analysis,

$$u_x = v_y \quad (a) \qquad u_y = -v_x \quad (b) \tag{3}$$

(where the variable *t* has here been renamed *y*) constitute an auto-BT for the *Laplace equation*,

$$P\,[w] \equiv w_{xx} + w_{yy} = 0 \tag{4}$$

Let us explain this: Suppose we want to solve the system (3) for *u*, for a given choice of the function $v(x,y)$. To see if the PDEs (*a*) and (*b*) match for solution for *u*, we must compare them in some way. We thus differentiate (*a*) with respect to *y* and (*b*) with respect to *x*, and equate the mixed derivatives of *u*. That is, we apply the integrability condition $(u_x)_y = (u_y)_x$. In this way we eliminate the variable *u* and find the condition that must be obeyed by $v(x,y)$:

$$P\,[v] \equiv v_{xx} + v_{yy} = 0 \;\; .$$

Similarly, by using the integrability condition $(v_x)_y = (v_y)_x$ to eliminate *v* from the system (3), we find the necessary condition in order that this system be integrable for *v*, for a given function $u(x,y)$:

$$P\,[u] \equiv u_{xx} + u_{yy} = 0 \;\; .$$

In conclusion, the integrability of system (3) with respect to either variable u or v requires that the other variable must satisfy the Laplace equation (4).

Let now $v_0(x,y)$ be a known solution of the Laplace equation (4). Substituting $v=v_0$ in the system (3), we can integrate this system with respect to u. As can be shown by eliminating v_0 from the system, the solution u will also satisfy the Laplace equation (4). As an example, by choosing the solution $v_0(x,y)=xy$ we find a new solution $u(x,y)=(x^2-y^2)/2+C$.

2. The *Liouville equation* is written

$$P[u] \equiv u_{xt} - e^u = 0 \quad \Leftrightarrow \quad u_{xt} = e^u \tag{5}$$

Due to its nonlinearity, this PDE is hard to integrate directly. A solution is thus sought by means of a BT. We consider an auxiliary function $v(x,t)$ and an associated PDE,

$$Q[v] \equiv v_{xt} = 0 \tag{6}$$

We also consider the system of first-order PDEs,

$$u_x + v_x = \sqrt{2}\, e^{(u-v)/2} \quad (a) \qquad u_t - v_t = \sqrt{2}\, e^{(u+v)/2} \quad (b) \tag{7}$$

Differentiating the PDE (a) with respect to t and the PDE (b) with respect to x, and eliminating $(u_t - v_t)$ and (u_x+v_x) in the ensuing equations with the aid of (a) and (b), we find that u and v satisfy the PDEs (5) and (6), respectively. Thus, the system (7) is a BT connecting solutions of (5) and (6). Starting with the trivial solution $v=0$ of (6), and integrating the system (7), which reads

$$u_x = \sqrt{2}\, e^{u/2}\ , \quad u_t = \sqrt{2}\, e^{u/2} \tag{7a}$$

we find a nontrivial solution of (5):

$$u(x,t) = -2 \ln\left(C - \frac{x+t}{\sqrt{2}} \right)$$

(see Appendix).

3. The *"sine-Gordon" equation* has applications in various areas of Physics, e.g., in the study of crystalline solids, in the transmission of elastic waves, in magnetism, in elementary-particle models, etc. The equation (whose name is a pun on the related linear Klein-Gordon equation) is written

$$P[u] \equiv u_{xt} - \sin u = 0 \quad \Leftrightarrow \quad u_{xt} = \sin u \tag{8}$$

The following system of equations is an auto-BT for the nonlinear PDE (8):

$$\frac{1}{2}(u+v)_x = a \sin\left(\frac{u-v}{2}\right)\ , \qquad \frac{1}{2}(u-v)_t = \frac{1}{a} \sin\left(\frac{u+v}{2}\right) \tag{9}$$

where a ($\neq 0$) is an arbitrary real constant. [Because of the presence of a, the system (9) is called a *parametric* BT.] When u is a solution of (8) the BT (9) is integrable for v, which, in turn, also is a solution of (8): $P[v]=0$; and vice versa. Starting with the trivial solution $v=0$ of $v_{xt}=\sin v$, and integrating the system (9), which reads

$$u_x = 2a \sin\frac{u}{2} \quad , \quad u_t = \frac{2}{a}\sin\frac{u}{2} \tag{9a}$$

we obtain a new solution of (8):

$$u(x,t) = 4\arctan\left\{ C\exp\left(a x + \frac{t}{a} \right) \right\}$$

(see Appendix).

3. Method of parametric conjugate solutions

As presented in the previous section, a BT is an auxiliary device for constructing solutions of a (usually nonlinear) PDE from known solutions of the same or another PDE. The related problem where solutions of the differential system representing the BT itself are sought is also of interest, however, and has been studied in connection with the Maxwell equations of electromagnetism [7,8].

To be specific, assume that we need to integrate a given system of PDEs connecting two unknown functions $u(x,y)$ and $v(x,y)$:

$$B_i[u,v]=0 \quad , \quad i=1,2 \tag{10}$$

Suppose that the integrability of the above system for both functions requires that u and v separately satisfy the respective PDEs

$$P[u]=0 \quad (a) \qquad Q[v]=0 \quad (b) \tag{11}$$

That is, the system (10) is a BT connecting solutions of the PDEs (11). Assume, now, that these PDEs possess known *parameter-dependent solutions* of the form

$$u = f(x,y;\alpha,\beta,\ldots) \quad , \qquad v = g(x,y;\kappa,\lambda,\ldots) \tag{12}$$

where α, β, κ, λ, etc., are (real or complex) parameters. If values of these parameters can be determined for which u and v jointly satisfy the system (10), we say that the solutions u and v of the PDEs (11a) and (11b), respectively, are *conjugate through the BT* (10) (or *BT-conjugate*, for short). By finding a pair of BT-conjugate solutions (12) one thus automatically obtains a solution of the system (10).

Note that solutions of *both* integrability conditions (11) of the system (10) must now be known in advance! From the practical point of view the method is thus most applicable in *linear* problems, since it is much easier to find parameter-dependent solutions of the PDEs (11) in this case.

Let us see an example: Going back to the Cauchy-Riemann relations (3), which is an auto-BT connecting solutions of the Laplace equation (4), we try the following parametric solutions of the latter PDE:

$$u(x,y) = \alpha(x^2 - y^2) + \beta x + \gamma y \ ,$$
$$v(x,y) = \kappa xy + \lambda x + \mu y \ .$$

Substituting these expressions into the BT (3), we find that $\kappa=2\alpha$, $\mu=\beta$ and $\lambda=-\gamma$. Therefore, the solutions

$$u(x,y) = \alpha(x^2 - y^2) + \beta x + \gamma y \ ,$$
$$v(x,y) = 2\alpha xy - \gamma x + \beta y$$

of the Laplace equation are BT-conjugate through the Cauchy-Riemann relations.

As a counter-example, let us try a different combination of parametric solutions:

$$u(x,y) = \alpha xy \ , \quad v(x,y) = \beta xy \ .$$

Inserting these into the system (3) and taking into account the independence of x and y, we find that the only possible values of the parameters α and β are $\alpha=\beta=0$, so that $u(x,y)= v(x,y)=0$. Thus, no non-trivial BT-conjugate solutions exist in this case.

4. BTs as recursion operators for symmetries of PDEs

The concept of symmetries of PDEs has been extensively discussed in [1] and [9]. Let us review the main ideas:

Consider a PDE $F[u]=0$, where $u=u(x,t)$. A transformation $u(x,t) \to u'(x,t)$ from the function u to a new function u' represents a *symmetry* of this PDE if the following condition is satisfied: $u'(x,t)$ is a solution of $F[u]=0$ *if* $u(x,t)$ is a solution. That is,

$$F[u'] = 0 \quad \text{when} \quad F[u] = 0 \qquad (13)$$

An *infinitesimal symmetry transformation* is written

$$u' = u + \delta u = u + \alpha Q[u] \qquad (14)$$

where α is an infinitesimal parameter. The function $Q[u] \equiv Q(x, t, u, u_x, u_t, ...)$ is called the *symmetry characteristic* of the transformation (14).

In order that a function $Q[u]$ be a symmetry characteristic for the PDE $F[u]=0$, it must satisfy a certain PDE that expresses the *symmetry condition* for $F[u]=0$. We write, symbolically,

$$S(Q;u) = 0 \quad \text{when} \quad F[u] = 0 \qquad (15)$$

where the expression S depends *linearly* on Q and its partial derivatives. Thus, (15) is a linear PDE for Q, in which equation the variable u enters as a sort of parametric function that is required to satisfy the PDE $F[u]=0$.

A *recursion operator* $\hat{R}$ [10] is a linear operator which, acting on any symmetry characteristic Q, produces a new symmetry characteristic $Q' = \hat{R}Q$. That is,

$$S(\hat{R}Q;u) = 0 \quad \text{when} \quad S(Q;u) = 0 \qquad (16)$$

It is easy to show that *any power of a recursion operator also is a recursion operator*. This means that, starting with any symmetry characteristic Q, one may in principle obtain an infinite set of characteristics (thus, an infinite number of symmetries) by repeated application of the recursion operator.

A new approach to recursion operators was suggested in the early 1990s [2,3] (see also [4-6] and [11-13]). According to this view, a recursion operator for the PDE $F[u]=0$ is an auto-BT for the linear PDE (15) that expresses the symmetry condition of $F[u]=0$; that is, a BT producing new solutions Q' of (15) from old ones, Q. Typically, this type of BT produces *nonlocal* symmetries, i.e., symmetry characteristics depending on *integrals* (rather than derivatives) of u.

As an example, consider the *chiral field equation*

$$F[g] \equiv (g^{-1}g_x)_x + (g^{-1}g_t)_t = 0 \tag{17}$$

(as usual, subscripts denote partial differentiations) where g is a $GL(n,C)$-valued function of x and t (i.e., an invertible complex $n\times n$ matrix, differentiable for all x, t).

Let $Q[g]$ be a symmetry characteristic of the PDE (17). It is convenient to put

$$Q[g] = g\,\Phi[g]$$

and write the corresponding infinitesimal symmetry transformation in the form

$$g' = g + \delta g = g + \alpha g\,\Phi[g] \tag{18}$$

The symmetry condition that Q must satisfy will be a PDE linear in Q, thus in Φ also. As can be shown [9] this PDE is

$$S(\Phi; g) \equiv \Phi_{xx} + \Phi_{tt} + [g^{-1}g_x,\, \Phi_x] + [g^{-1}g_t,\, \Phi_t] = 0 \tag{19}$$

which must be valid when $F[g]=0$ (where, in general, $[A, B] \equiv AB-BA$ denotes the commutator of two matrices A and B).

For a given g satisfying $F[g]=0$, consider now the following system of PDEs for the matrix functions Φ and Φ':

$$\begin{aligned} \Phi'_x &= \Phi_t + [g^{-1}g_t\,,\, \Phi] \\ -\Phi'_t &= \Phi_x + [g^{-1}g_x\,,\, \Phi] \end{aligned} \tag{20}$$

The integrability condition $(\Phi'_x)_t = (\Phi'_t)_x$, together with the equation $F[g]=0$, require that Φ be a solution of (19): $S(\Phi\,; g) = 0$. Similarly, by the integrability condition $(\Phi_t)_x = (\Phi_x)_t$ one finds, after a lengthy calculation: $S(\Phi'; g) = 0$.

In conclusion, for any g satisfying the PDE (17), the system (20) is a BT relating solutions Φ and Φ' of the symmetry condition (19) of this PDE; that is, relating different symmetries of the chiral field equation (17). Thus, if a symmetry characteristic $Q=g\Phi$ of (17) is known, a new characteristic $Q'=g\Phi'$ may be found by integrating the BT (20); the converse is also true. Since the BT (20) produces new symmetries from old ones, it may be regarded as a *recursion operator* for the PDE (17).

As an example, for any constant matrix M the choice $\Phi=M$ clearly satisfies the symmetry condition (19). This corresponds to the symmetry characteristic $Q=gM$. By integrating the BT (20) for Φ', we get $\Phi'=[X, M]$ and $Q'=g[X, M]$, where X is the "potential" of the PDE (17), defined by the system of PDEs

$$X_x = g^{-1} g_t \ , \quad -X_t = g^{-1} g_x \tag{21}$$

Note the *nonlocal* character of the BT-produced symmetry Q', due to the presence of the potential X. Indeed, as seen from (21), in order to find X one has to *integrate* the chiral field g with respect to the independent variables x and t. The above process can be continued indefinitely by repeated application of the recursion operator (20), leading to an infinite sequence of increasingly nonlocal symmetries.

Appendix

We describe the process of integrating the BTs (7*a*) and (9*a*) for the Liouville equation and the sine-Gordon equation, respectively.

1. The system (7*a*) reads

$$u_x = \sqrt{2}\ e^{u/2} \tag{A.1}$$

$$u_t = \sqrt{2}\ e^{u/2} \tag{A.2}$$

We integrate (A.1) for x, treating t as constant:

$$\frac{du}{dx} = \sqrt{2}\ e^{u/2} \ \Rightarrow\ \int e^{-u/2} du = \sqrt{2} \int dx \ \Rightarrow\ e^{-u/2} = -\frac{x}{\sqrt{2}} + h(t)$$

[where $h(t)$ is a function to be determined], from which we have that

$$u = -2\ln\left[-\frac{x}{\sqrt{2}} + h(t)\right] \quad \text{and therefore} \quad u_t = \frac{-2h'(t)}{-\frac{x}{\sqrt{2}} + h(t)} \ .$$

Substituting the above results into (A.2), we get:

$$h'(t) = -\frac{1}{\sqrt{2}} \quad \Rightarrow \quad h(t) = -\frac{t}{\sqrt{2}} + C \ .$$

Thus we finally have:

$$u(x,t) = -2\ln\left(C - \frac{x+t}{\sqrt{2}}\right) .$$

2. The system (9*a*) reads

$$u_x = 2a \sin\frac{u}{2} \qquad \text{(A.3)}$$

$$u_t = \frac{2}{a} \sin\frac{u}{2} \qquad \text{(A.4)}$$

Integrating (A.3) for x and using the integral formula

$$\int \frac{du}{\sin ku} = \frac{1}{k} \ln\left(\tan\frac{ku}{2} \right)$$

we have:

$$\frac{du}{dx} = 2a \sin\frac{u}{2} \quad \Rightarrow \quad \int \frac{du}{\sin(u/2)} = 2a \int dx \quad \Rightarrow$$

$$\ln\left(\tan\frac{u}{4} \right) = ax + g(t) \qquad \text{(A.5)}$$

Similarly, integrating (A.4) for t we find:

$$\ln\left(\tan\frac{u}{4} \right) = \frac{t}{a} + h(x) \qquad \text{(A.6)}$$

By comparing (A.5) and (A.6) we have that

$$ax + g(t) = \frac{t}{a} + h(x) \quad \Rightarrow \quad h(x) - ax = g(t) - \frac{t}{a} \ .$$

But, a function of x cannot be identically equal to a function of t unless both are equal to the same constant *C*: $h(x) - ax = g(t) - t/a = C \ \Rightarrow$

$$h(x) = ax + C \ , \quad g(t) = \frac{t}{a} + C \ .$$

From (A.5) and (A.6) we then get

$$\ln\left(\tan\frac{u}{4} \right) = ax + \frac{t}{a} + C \quad \Rightarrow \quad (\text{ by putting } C \text{ in place of } e^C \ )$$

$$u(x,t) = 4 \arctan\left\{ C \exp\left(ax + \frac{t}{a} \right) \right\} \ .$$

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