C. J. PAPACHRISTOU

ELEMENTS OF
MATHEMATICAL ANALYSIS

An Informal Introduction
for Physics and Engineering Students

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PREFACE

This short textbook is by no means a complete book on mathematical analysis. It is basically a concise, informal introduction to differentiation and integration of real functions of a single variable, supplemented with an elementary discussion of first-order differential equations, an introduction to differentiation and integration in higher dimensions, and an introduction to complex analysis. Functional series (and, in particular, power series) are also discussed. The book may serve as a tutorial resource in a short-term introductory course of mathematical analysis for beginning students of physics and engineering who need to use differential and integral calculus primarily for applications.

Having taught introductory Physics at the Hellenic Naval Academy for over three decades, I have often experienced situations where my first-year undergraduates needed reinforcement of their background in advanced calculus in order to properly follow the Physics course from the outset. This need led to the idea of writing a short, practical handbook that would be especially useful for self-study “in a hurry”. The present textbook is a translated and expanded version of the author’s lecture notes written originally in Greek. Proofs of theoretical statements are limited to those considered pedagogically useful, while the theory is amply supplemented with carefully chosen examples. For a deeper study of the subject the reader is referred to the bibliography cited at the end of the book.

Despite the essentially practical character of the book, proper attention is given to conceptual subtleties inherent in the subject. In particular, the concept of the differential of a function is carefully examined and its relation to the “differential” inside an integral is explained. For pedagogical purposes the discussion of the indefinite integral – as an infinite collection of antiderivatives – precedes that of the definite integral; it is shown, however, that the latter concept leads in a natural way to the former by allowing variable limits of integration.

The Appendix contains useful mathematical formulas and properties needed for the exercises, as well as a more detailed discussion of the concept of continuity of a function and its relationship with differentiability. Finally, answers to selected exercises are provided.

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CHAPTER 1

FUNCTIONS

1.1 Real Numbers

There are various sets of numbers in mathematics, such as the set of *natural numbers*, $N=\{1,2,3,\ldots\}$, the set of *integers*, $Z=\{0, \pm1, \pm2, \pm3,\ldots\}$, and the set of *rational numbers*, $Q=\{\, p/q$, where p, q are integers and $q\neq 0\}$. Numbers such as $\sqrt{2}, \sqrt{3}, \ \ln 3$, etc., which *cannot* be expressed as quotients p/q of integers, are called *irrational*. Rational and irrational numbers together constitute the set of *real numbers*, R.

In the set R of real numbers one may define various types of *intervals*:

Open interval:	$(a, b) = \{\, x \,/\, x\in R,\ a < x < b \,\}$
Closed interval:	$[a, b] = \{\, x \,/\, x\in R,\ a \leq x \leq b \,\}$
Semi-closed intervals:	$[a, b) = \{\, x \,/\, x\in R,\ a \leq x < b \,\}$ $(a, b] = \{\, x \,/\, x\in R,\ a < x \leq b \,\}$
Infinite intervals:	$(-\infty, c), (c, +\infty), (-\infty, c], [c, +\infty), (-\infty, +\infty)$

1.2 Functions

Let $D\subseteq R$ be a subset of R. We consider a rule $f: D \to R$, such that, to every element $x\in D$ there corresponds a *unique* element $y\in R$ (two or more elements of D *may*, however, correspond to the same element of R). We write:

$$(x\in D) \overset{f}{\mapsto} (y\in R) \quad \text{or} \quad y = f(x) .$$

The rule f constitutes a *real function*. We say that the *dependent variable* y is a function of the *independent variable* x. The set D is called the *domain of definition* of f, while the set $\{\, y= f(x) \,/\, x\in D\}\equiv f(D)$ is called the *range* of f.

Given a function $y= f(x)$ we can draw the corresponding *graph* (Fig. 1.1). We assume that the quantities x and y are *dimensionless* and, moreover, equal lengths on the x- and y-axes correspond to *equal changes* of x and y.

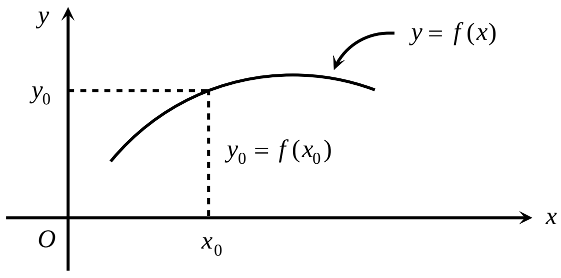


Fig. 1.1. Graph of a function.

A function $y=f(x)$ is said to be *continuous* at the point $x=x_0$ if its value $y_0=f(x_0)$ at that point is defined and is equal to the limit of $f(x)$ as $x\to x_0$:

$$\lim_{x\to x_0} f(x) = f(x_0)\ .$$

In practical terms we may say that the graph of $f(x)$ is a continuous curve at $x=x_0$ (it does not "break" into two separate curves at this point). If we set $x-x_0=\Delta x$ and $f(x)-f(x_0)=y-y_0=\Delta y$, then, by the definition of a continuous function it follows that $\Delta y\to 0$ when $\Delta x\to 0$. More on continuous functions can be found in the Appendix.

Below is a list of some elementary functions:

Constant function:	$y = f(x) = c \quad (c \in R)$
Power function:	$y = f(x) = x^a \quad (a \in R)$
Exponential function:	$y = f(x) = e^x$
Logarithmic function:	$y = f(x) = \ln x$
Trigonometric functions:	$y = f(x) = \sin x,\ \cos x,\ \tan x,\ \cot x$
Inverse trigonometric functions:	$y = f(x) = \mathrm{arc}\sin x,\ \mathrm{arc}\cos x,\ \mathrm{arc}\tan x,\ \mathrm{arc}\cot x$

By combining elementary functions we can construct composite functions. Let us consider the functions $y=g(u)$ and $u=h(x)$. We write

$$y = g[h(x)] \equiv (g\circ h)(x)\ .$$

We thus define the *composite function* $f = g\circ h$, so that

$$y = f(x) = g[h(x)] \equiv (g\circ h)(x)\ .$$

To simplify our notation we may write $y=y(x)$ instead of the more explicit $y=f(x)$. Similarly, $y=y(u)$ and $u=u(x)$. Then,

$$y = y(x) \quad \Leftrightarrow \quad [\, y = y(u),\ \ u = u(x)\,]\ .$$

Examples:

1. The composite function $y = y(x) = e^{\sqrt{x}}$ can be decomposed into simple ones, as follows:

$$y = y(u) = e^u\ , \quad u = u(x) = \sqrt{x} = x^{1/2}$$

while the function $y = y(x) = e^{\sqrt{x^2+1}}$ is decomposed as

$$y = y(u) = e^u\ , \quad u = u(w) = \sqrt{w} = w^{1/2}\ , \quad w = w(x) = x^2+1\ .$$

2. The function $y = y(x) = \ln(1+\sin^2 x)$ is decomposed as follows:

$$y = y(u) = \ln u \ , \quad u = u(w) = 1 + w^2 , \quad w = w(x) = \sin x \ .$$

3. For the function $y = y(x) = \cos^3 \sqrt{x^6+1}$ we write

$$y = y(u) = u^3 , \quad u = u(w) = \cos w , \quad w = w(z) = \sqrt{z} = z^{1/2} , \quad z = z(x) = x^6 + 1 \ .$$

1.3 Domain of Definition of a Function

Consider a function $y=f(x)$. Its *domain of definition*, D, is the largest subset of R for which $y \in R$, $\forall x \in D$. Practically this means that the values y of $f(x)$ are *real* and *finite* for all $x \in D$. Below are the domains of definition of some elementary functions:

$$y = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \quad | \quad D = R = (-\infty, +\infty)$$
$$y = f(x) = \frac{1}{x} \quad | \quad D = R - \{0\} = (-\infty, 0) \cup (0, +\infty)$$
$$y = f(x) = \sqrt{x} \quad | \quad D = [0, +\infty)$$
$$y = f(x) = e^x \quad | \quad D = R = (-\infty, +\infty)$$
$$y = f(x) = \ln x \quad | \quad D = (0, +\infty)$$
$$y = f(x) = \sin x, \cos x \quad | \quad D = R = (-\infty, +\infty)$$
$$y = f(x) = \tan x \quad | \quad D = R - \{k\pi + \pi/2 , \ k = 0, \pm 1, \pm 2, \cdots\}$$
$$y = f(x) = \cot x \quad | \quad D = R - \{k\pi , \ k = 0, \pm 1, \pm 2, \cdots\}$$
$$y = f(x) = \arcsin x, \arccos x \quad | \quad D = [-1, 1]$$
$$y = f(x) = \arctan x, \operatorname{arccot} x \quad | \quad D = R = (-\infty, +\infty)$$

Let us also see some examples of domains of composite functions:

$$y = \frac{1}{\sqrt{x}} \quad \Rightarrow \quad y = \frac{1}{u} \ , \ u = \sqrt{x} \quad | \quad D = (0, +\infty)$$
$$y = \frac{1}{\ln x} \quad \Rightarrow \quad y = \frac{1}{u} \ , \ u = \ln x \quad | \quad D = (0, +\infty) - \{1\} = (0,1) \cup (1, +\infty)$$
$$y = \frac{1}{\sqrt{\ln x}} \quad \Rightarrow \quad y = \frac{1}{u} \ , \ u = \sqrt{w} \ , \ w = \ln x \quad | \quad D = (1, +\infty)$$

Exercise 1.1 Find the domains of definition of the following functions:

(1) $y = \dfrac{\ln(x^2+1)}{\sqrt{x^6+1}}$ (2) $y = \sqrt{1-x^2}$ (3) $y = \dfrac{\arctan x}{\sqrt{x^2-1}}$ (4) $y = \arccos\left(\dfrac{2x}{x+1}\right)$

(5) $y = \dfrac{\ln(x+5)}{\sqrt{8-x^3}}$ [Hint : $a^3-b^3=(a-b)(a^2+ab+b^2)$]

(6) $y = \ln(\ln x)$ (7) $y = \tan 2x$ (8) $y = \tan\dfrac{x}{3}$

1.4 Implicit and Multiple-Valued Functions

Implicit functions are expressions of the form

$$F(x, y) = 0 \tag{1}$$

which relate the variables x and y without expressing y in terms of x directly. In the special case where $F(x, y) = f(x) - y$, relation (1) yields a function of the standard (explicit) form $y = f(x)$.

Examples:

$$F(x, y) \equiv y^3 - 3xy + x^3 = 0$$

$$F(x, y) \equiv y + x e^y - 1 = 0$$

The functions we have met so far were *single-valued*, in the sense that to every value of $x \in D$ there corresponds a *unique* value of $y = f(x)$. A function that does not conform to this restriction is called *multiple-valued*. In general, implicit functions are multiple-valued.

Example:

$$x^2 + y^2 = 1 \iff F(x, y) \equiv x^2 + y^2 - 1 = 0 .$$

The graph is the unit circle on the plane (Fig. 1.2). We write:

$$y = \pm(1-x^2)^{1/2} \mid D = [-1, 1] .$$

We notice that to every value of $x \in D$ there correspond *two* values of y.

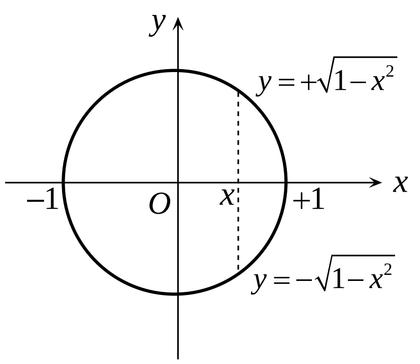


Fig. 1.2. A unit circle on the plane.

1.5 Exponential and Logarithmic Functions

We consider the sequence

$$a_n = \left(1+\frac{1}{n}\right)^n \quad , \quad n = 1,2,3,\cdots$$

We define:

$$e = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \cong 2.7$$

Definition: Let $a>0$ be a positive real number, and let

$$a = e^b$$

for some $b \in R$. The number

$$b = \ln a$$

is called the *logarithm* of *a*. Notice that we *cannot* define $\ln a$ for $a \le 0$! Moreover,

$$\ln a = \ln c \iff a = c$$

Examples:

1. $\ln 1 = ?$

Let $\ln 1 = x$. Then, $1 = e^x \Rightarrow e^x = e^0 \Rightarrow x=0 \Rightarrow$ $\boxed{\ln 1 = 0}$

2. $\ln e = ?$

Let $\ln e = x$. Then, $e = e^x \Rightarrow e^x = e^1 \Rightarrow x=1 \Rightarrow$ $\boxed{\ln e = 1}$

3. $\ln (1/e) = ?$

Let $\ln (1/e) = x$. Then, $1/e = e^x \Rightarrow e^x = e^{-1} \Rightarrow x= -1 \Rightarrow$ $\boxed{\ln (1/e) = -1}$

4. Similarly we can show that $\boxed{\ln\sqrt{e} = 1/2 \ , \quad \ln(1/\sqrt{e}) = -1/2}$

5. $\lim_{x\to 0^+} \ln x = ?$

In general, $\ln x = y \iff x= e^y$. We notice that $x\to 0^+$ as $y \to -\infty$. Thus, conversely, $y= \ln x \to -\infty$ when $x\to 0^+$. That is,

$$\boxed{\lim_{x\to 0^+} \ln x = -\infty}$$

Properties of logarithms:

1. $\ln(e^a) = a\ ,\quad \forall a \in R$
2. $e^{\ln a} = a\ ,\quad \forall a \in R^+$
3. $\ln(ab) = \ln a + \ln b \quad (a > 0,\ b > 0)$
4. $\ln\dfrac{a}{b} = \ln a - \ln b = -\ln\dfrac{b}{a} \quad (a > 0,\ b > 0)$
5. $\ln\dfrac{1}{a} = -\ln a \quad (a > 0)$
6. $\ln(a^k) = k\ln a \quad (a > 0,\ k \in R)$

Proof :

1. Let $\ln(e^a) = x \ \Rightarrow\ e^a = e^x \ \Rightarrow\ x = a$.
2. Let $e^{\ln a} = x \ \Rightarrow\ \ln a = \ln x \ \Rightarrow\ x = a$.
3. Let $\ln a = x$, $\ln b = y$, $\ln(a\,b) = z$. We show that $x+y = z$:

   $\ln(ab) = z \ \Rightarrow\ ab = e^z ,\quad \ln a = x \ \Rightarrow\ a = e^x ,\quad \ln b = y \ \Rightarrow\ b = e^y$;

   $ab = e^z \ \Rightarrow\ e^x e^y = e^z \ \Rightarrow\ e^{x+y} = e^z \ \Rightarrow\ x + y = z$.
4. Let $\ln a = x$, $\ln b = y$, $\ln(a/b) = z$. We show that $x-y = z$:

   $\ln(a/b) = z \ \Rightarrow\ a/b = e^z ,\quad \ln a = x \ \Rightarrow\ a = e^x ,\quad \ln b = y \ \Rightarrow\ b = e^y$;

   $a/b = e^z \ \Rightarrow\ e^x/e^y = e^z \ \Rightarrow\ e^{x-y} = e^z \ \Rightarrow\ x - y = z$.

   $\ln(a/b) = \ln a - \ln b = -(\ln b - \ln a) = -\ln(b/a)$.
5. $\ln(1/a) = \ln 1 - \ln a = 0 - \ln a = -\ln a$.
6. Let $\ln(a^k) = x$, $\ln a = y$. We show that $x = ky$:

   $\ln(a^k) = x \ \Rightarrow\ a^k = e^x ,\quad \ln a = y \ \Rightarrow\ a = e^y$;

   $a^k = e^x \ \Rightarrow\ (e^y)^k = e^x \ \Rightarrow\ e^{ky} = e^x \ \Rightarrow\ ky = x$.

Exercise 1.2 Show that

$$\ln\left(\frac{a\,b}{c}\right) = \ln a + \ln b - \ln c$$

Exercise 1.3 Find the values of the following expressions:

(1) $\ln\left(\sin\dfrac{\pi}{2}\right)$ (2) $\ln\left(\dfrac{1}{e^2}\right)$ (3) $\ln\left(\dfrac{e^2\sqrt{e}}{\sqrt{e^3}}\right)$

The function

$$y = f(x) = e^x \equiv \exp x$$

is called the *exponential function* and is defined for all $x \in R$. Its domain of definition is, therefore, $D = R$.

The function

$$y = f(x) = \ln x$$

is called the *logarithmic function*. What is its domain of definition? We notice that

$$y = \ln x \;\Rightarrow\; x = e^y \;\Rightarrow\; x > 0 , \;\forall y \in R \;.$$

Hence, $D = R^+ = (0\,,\, +\infty)$. As we showed earlier, $\lim\limits_{x \to 0^+} \ln x = -\infty$.

Graphs:

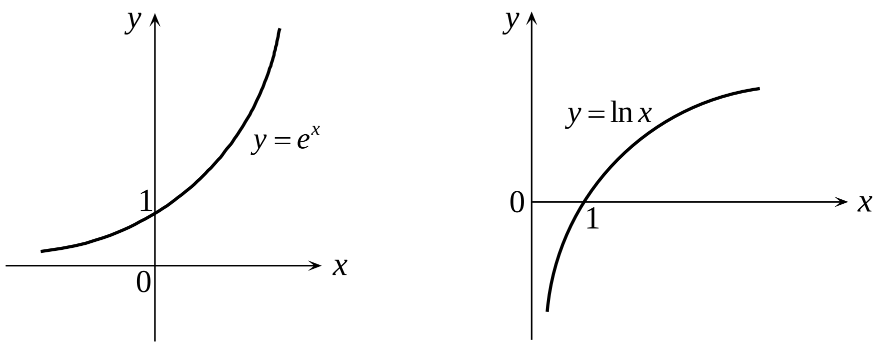


Fig. 1.3. Graphs of exponential and logarithmic functions.

1.6 Linear Function

The function

$$y = f(x) = a\,x + b \qquad (a \neq 0) \tag{1}$$

is called *linear function* because its graph is a straight line (Fig. 1.4).

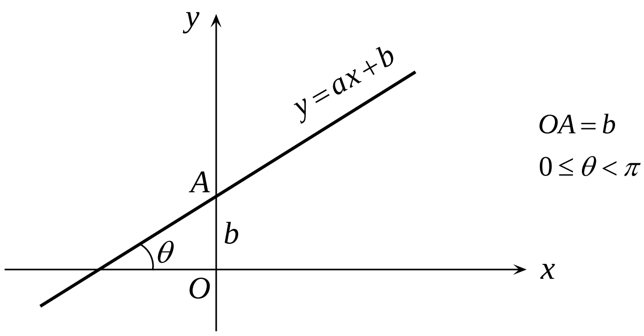


Fig. 1.4. Graph of a linear function.

For $x=0$ we have that $f(0)=b$. The geometrical significance of a is found as follows (see Fig. 1.5).

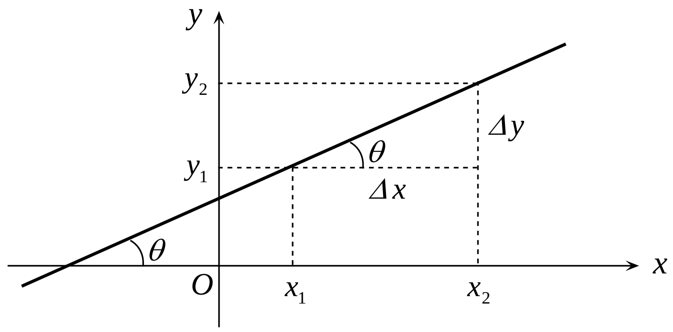


Fig. 1.5. Graph of linear function.

Let $y_1=ax_1+b$, $y_2=ax_2+b$. Subtracting the first equation from the second and setting $\Delta x=x_2-x_1$, $\Delta y=y_2-y_1$, we find that

$$\Delta y = a\,\Delta x \quad \Leftrightarrow \quad \frac{\Delta y}{\Delta x} = a \equiv const. \tag{2}$$

Relation (2) is the necessary *and* sufficient condition in order that the function $y=f(x)$ be linear. Now, from the above figure we see that $\Delta y/\Delta x=\tan\theta$. Hence,

$$a = \tan\theta \tag{3}$$

The constant a is called the *slope* of the straight line (1).

Problem: Find the equation of a line passing through the point (x_0, y_0) and forming an angle θ with the x-axis.

Solution: By (2) and (3) we have that $\Delta y= a\Delta x$, where $a= \tan\theta$, $\Delta x=x-x_0$ and $\Delta y = y-y_0$. Thus,

$$y - y_0 = a\,(x - x_0) \tag{4}$$

Alternatively, we seek an equation of the form (1) for suitable values of a and b. The constant a is equal to $\tan\theta$. Putting $x= x_0$ and $y= y_0$ in (1), we have: $y_0= a\,x_0+b \Rightarrow b= y_0-ax_0$. Substituting this value of b into (1), we get (4).

Problem: Find the equation of a line passing through the points (x_1, y_1) and (x_2, y_2).

Solution: Since the line passes through (x_1, y_1) it will be described by an equation of the form (4) with (x_1, y_1) in place of (x_0, y_0): $y-y_1= a\,(x-x_1)$. On the other hand, the slope a is equal to $a= \Delta y/\Delta x= (y_2-y_1)/(x_2-x_1)$. We thus have:

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \tag{5}$$

By a property of proportions (see Appendix), from (5) $\Rightarrow (y-y_1)/(x-x_1)=(y-y_2)/(x-x_2)$ (show this!). We thus obtain an equation equivalent to (5).

1.7 Quadratic Function

The function

$$y = f(x) = a x^2 + b x + c \qquad (a \neq 0)$$

is called *quadratic function* and is represented graphically by a *parabola* (Fig. 1.6).

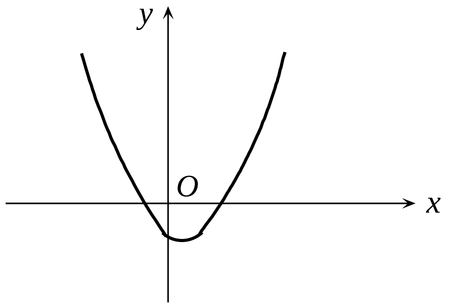


Fig. 1.6. A parabola.

The *roots* of a quadratic function are the real or complex numbers ρ_1 , ρ_2 for which $f(\rho_1)= f(\rho_2)=0$; they are given by the formula

$$\rho_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} \quad , \quad \Delta = b^2 - 4ac \, .$$

The roots are real and different if $\Delta>0$, real and equal if $\Delta=0$, and complex conjugates if $\Delta<0$.

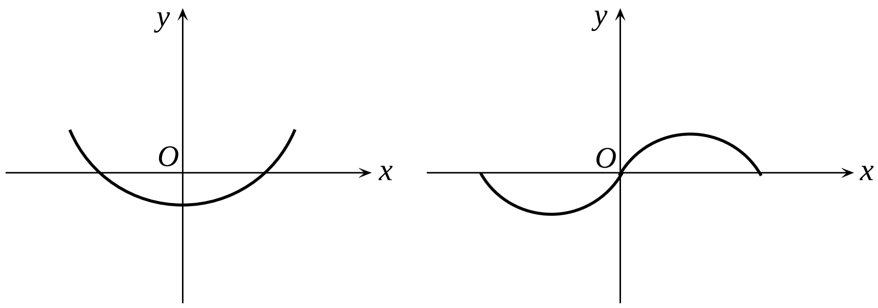
1.8 Even and Odd Functions

Consider a function $y= f(x)$ with domain of definition D. We assume that if $x\in D$, then $(-x)\in D$ also. We say that

> $f(x)$ is an *even* function if $f(-x) = f(x)$, $\forall x\in D$, while
>
> $f(x)$ is an *odd* function if $f(-x) = -f(x)$, $\forall x\in D$.

Of course, an arbitrary function need be neither even nor odd! For example, the function $f(x)=x^3+1$ is neither even nor odd.

The graph of an *odd* function (see Fig. 1.7) always passes through the origin of the x-y system of axes (provided, of course, that the value $x=0$ belongs to D). Indeed, by putting $x=0$ in the relation $f(-x)+ f(x)=0$ we find that $f(0)=0$.



Even function *Odd function*

Fig. 1.7. An even and an odd function.

Examples:

<u>Even</u>	<u>Odd</u>
$f(x) = c,\ x^2,\ x^4,\ x^6,\ \cdots$	$f(x) = x,\ x^3,\ x^5,\ x^7,\ \cdots$
$f(x) = \lvert x \rvert$	$f(x) = \sin x$
$f(x) = \cos x$	$f(x) = \tan x,\ \cot x$
$f(x) = e^x + e^{-x}$	$f(x) = e^x - e^{-x}$

Exercise 1.4 Prove the validity of the following statements:

- The product (and likewise the quotient) of two *even* or two *odd* functions is an *even* function.
- The product (and likewise the quotient) of an *even* and an *odd* function is an *odd* function.
- The sum and the difference of two *even* functions is an *even* function.
- The sum and the difference of two *odd* functions is an *odd* function.
- The sum of an even and an odd function is a function that is neither even nor odd.

Proposition: *Every* function $f(x)$ can be written as the sum of an *even* function $A(x)$ and an *odd* function $B(x)$.

Proof: We write

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] \equiv A(x) + B(x)\ , \ \text{where}$$

$$A(x) = \frac{1}{2}[f(x) + f(-x)]\ , \quad B(x) = \frac{1}{2}[f(x) - f(-x)]$$

It is not hard to show that $A(-x) = A(x)$ and $B(-x) = -B(x)$.

Example: For $f(x) = e^x$ we write:

$$e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) \equiv A(x) + B(x)\,.$$

Exercise 1.5 For each of the following functions, examine whether it is even, odd or neither.

(1) $f(x) = 2x^4 - 3x^2 + 1$ (2) $f(x) = 2x^3 - 3x$ (3) $f(x) = x^5 + 1$

(4) $f(x) = |x+1| + |x-1|$ (5) $f(x) = |x+1| - |x-1|$ (6) $f(x) = \ln\left|\dfrac{x-1}{x+1}\right|$

(7) $f(x) = x^3 \sin x$ (8) $f(x) = x^3 \cos x$ (9) $f(x) = \dfrac{\tan x}{x^5}$ (10) $f(x) = \dfrac{\cot x}{x^6}$

1.9 Periodic Functions

A function $y = f(x)$ is called *periodic* with *period* $a \neq 0$ (Fig. 1.8) if

$$f(x+a) = f(x) \tag{1}$$

If (1) is valid then it is true that, more generally,

$$f(x+ka) = f(x), \quad k = \pm 1, \pm 2, \pm 3, \dots$$

(show this!). That is, if a is a period of $f(x)$, then so is ka, where k is any integer. Typically, by "period" we mean the *smallest positive period* of a periodic function.

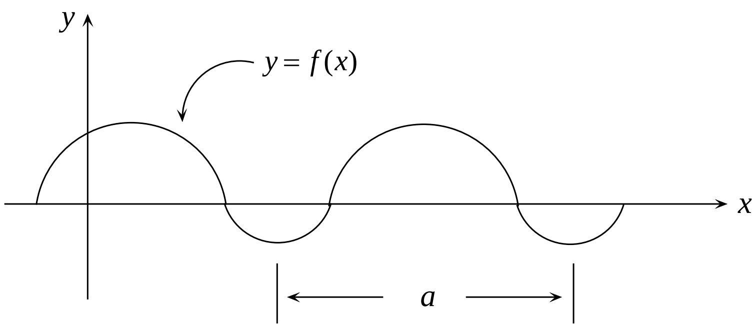


Fig. 1.8. Periodic function.

Examples:

In the following examples, use will be made of the trigonometric equations presented in the Appendix.

1. $y = f(x) = \sin x$. We check if f is periodic with period a:

$$f(x+a) = f(x) \;\Rightarrow\; \sin(x+a) = \sin x \;\Rightarrow\; x+a = x + 2k\pi \text{ or } x+a = (2k+1)\pi - x$$

so that $a = 2k\pi$ or $a = (2k+1)\pi - 2x$ ($k = 0, \pm 1, \pm 2, \pm 3, \dots$). The second solution is not acceptable since a must be a constant, independent of x. The solution $a = 2k\pi$ has a minimum positive value for $k = 1$. Therefore, $y = f(x) = \sin x$ is periodic with fundamental period equal to

$$a = 2\pi$$

2. $y=f(x)=\cos x$. Show that this function is periodic with period

$$a=2\pi$$

3. $y=f(x)=\sin 2x$ or $\cos 2x$. Show that these functions are periodic with period

$$a=\pi$$

4. $y=f(x)=\sin(x/2)$ or $\cos(x/2)$. Show that these functions are periodic with period

$$a=4\pi$$

5. $y=f(x)=\sin\lambda x$ or $\cos\lambda x$ $(\lambda\in R^{+})$. Show that these functions are periodic with period

$$a=2\pi/\lambda$$

6. $y=f(x)=\tan x$ or $\cot x$. Show that these functions are periodic with period

$$a=\pi$$

7. $y=f(x)=\tan\lambda x$ or $\cot\lambda x$ $(\lambda\in R^{+})$. Show that these functions are periodic with period

$$a=\pi/\lambda$$

8. Every constant function $y=f(x)=c$ is periodic with *arbitrary* period. Indeed: $f(x+a)=c=f(x)$, for any value of a.

Exercise 1.6 Show the following:

- If $f(x)$ is periodic with period a, then $\lambda f(x)$ and $f(x)+c$ (where λ, c are constants) will also be periodic with period a.
- Let $f_1(x)$ and $f_2(x)$ be periodic with period a. Then $f_1(x)\pm f_2(x)$ will also be periodic with period a.
- Let $f_1(x)$ and $f_2(x)$ be periodic with period a. Then $f_1(x)\cdot f_2(x)$ and $f_1(x)/f_2(x)$ will also be periodic with period a (which, however, may not be their *smallest* period).

Assume now that $f_1(x)$ and $f_2(x)$ are periodic with corresponding smallest periods a_1 and a_2. We want to check if the sum $f_1(x)+f_2(x)$ is a periodic function. This will be the case if $f_1(x)$ and $f_2(x)$ have some common period, not necessarily the smallest one of either $f_1(x)$ or $f_2(x)$. The sets of positive periods of the two functions are

$$S_1=\{ka_1 / k=1,2,3,\cdots\}=\{a_1, 2a_1, 3a_1, \cdots\},$$
$$S_2=\{ka_2 / k=1,2,3,\cdots\}=\{a_2, 2a_2, 3a_2, \cdots\}$$

Let us assume that the intersection of S_1 and S_2 is not the null set: $S_1\cap S_2\neq\varnothing$. Then the function $f_1(x)+f_2(x)$ will be periodic with period equal to the smallest element of $S_1\cap S_2$ (i.e., the least common multiple of a_1 and a_2).

How about the functions $f_1(x) \cdot f_2(x)$ and $f_1(x)/f_2(x)$? Again, the smallest element of the set $S_1 \cap S_2$ is a period of these functions, but it will not necessarily be their smallest period. Let us see an example:

We will check the periodicity of the function $f(x) = \tan x$. We can work in two ways:

(a) $f(x+a) = f(x) \Rightarrow \tan(x+a) = \tan x \Rightarrow x+a = x+k\pi \Rightarrow a = k\pi$ ($k=1,2,3,...$). The smallest value of the period is $a=\pi$.

(b) We write the given function in the form of a quotient: $f(x) = \sin x/\cos x$. The functions in both the numerator and the denominator are periodic with common (smallest) period 2π. This will also be a period for $f(x)$, but will it be its smallest period? Let a be the smallest period of $f(x)$. Then,

$$f(x+a) = f(x) \;\Rightarrow\; \frac{\sin(x+a)}{\cos(x+a)} = \frac{\sin x}{\cos x} \;.$$

This can be satisfied in either of two ways:

- $\sin(x+a) = \sin x$ *and* $\cos(x+a) = \cos x \;\Rightarrow\; x+a = x+2k\pi$, or
- $\sin(x+a) = -\sin x$ *and* $\cos(x+a) = -\cos x \;\Rightarrow\; x+a = x+(2k+1)\pi$.

Thus, $a = 2k\pi$ or $a=(2k+1)\pi$. These may be combined by writing $a=\lambda\pi$ ($\lambda=1,2,3,...$). The smallest value of a, for $\lambda=1$, is $a=\pi$. We notice that the period of the quotient $\sin x/\cos x$, equal to π, is *smaller* than the period 2π of both $\sin x$ and $\cos x$!

Examples:

1. Examine whether the functions $f(x) = \sin\sqrt{x}$ and $f(x) = \sin x^2$ are periodic.

Solution: In both cases the relation $f(x+a) = f(x)$ yields expressions for a that are not constant quantities but functions of x (show this). Therefore, neither of the given functions is periodic.

2. Examine the periodicity of the function $f(x) = 3\sin 2x + 2\cos 3x$.

Solution: Let $f_1(x) = 3\sin 2x$ and $f_2(x) = 2\cos 3x$. Then, $f(x) = f_1(x) + f_2(x)$. The function $f(x)$ will be periodic if the $f_1(x)$ and $f_2(x)$ have some common period; that is, if $S_1 \cap S_2 \neq \varnothing$, where S_1 and S_2 are the (infinite) sets of periods of the two functions. The period of $f(x)$ will then be the smallest element of the set $S_1 \cap S_2$. Now, we recall that the functions $\sin\lambda x$ and $\cos\lambda x$ are periodic with (smallest) period $2\pi/\lambda$. Thus the set of periods of each function is $2k\pi/\lambda$ ($k=1,2,3,...$). Analytically, for $\lambda=2$ and $\lambda=3$ we have:

$$S_1 = \{k\pi \;/\; k=1,2,3,\cdots\} = \{\pi, 2\pi, 3\pi, \cdots\} \;,$$
$$S_2 = \{\frac{2k\pi}{3} \;/\; k=1,2,3,\cdots\} = \{\frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi, \cdots\}$$

We observe that the smallest element of the intersection $S_1 \cap S_2$ is 2π. Hence the given function $f(x)$ is periodic with period $a=2\pi$.

3. Examine the periodicity of the function $f(x)=\sin^2 x$.

Solution: $f(x+a)=f(x) \Rightarrow \sin^2(x+a)=\sin^2 x$. This is satisfied in two ways:

- $\sin(x+a)=\sin x \;\Rightarrow\; x+a=x+2k\pi$ or $x+a=(2k+1)\pi-x$ (not acceptable),
- $\sin(x+a)=-\sin x \;\Rightarrow\; x+a=x+(2k+1)\pi$ or $x+a=2k\pi-x$ (not acceptable)

(the two solutions that were rejected would give an x-dependent a). We thus have that $a=2k\pi$ or $a=(2k+1)\pi$. Combining these results, we write: $a=\lambda\pi$ (λ=1,2,3,...). For the smallest period we set λ=1, so that $a=\pi$. The given function is thus periodic with period $a=\pi$.

4. We consider the functions

$$1,\; \cos\omega t,\; \sin\omega t,\; \cos 2\omega t,\; \sin 2\omega t, \;\ldots\;, \cos n\omega t,\; \sin n\omega t, \;\ldots$$

where ω is a positive constant. The constant function 1 is periodic with *arbitrary* period. The remaining functions have a common period $T=2\pi/\omega$ which, however, is the *smallest* period only for $\cos\omega t$ and $\sin\omega t$ (in general, $\cos n\omega t$ and $\sin n\omega t$ both have smallest period equal to $2\pi/n\omega=T/n$). We now consider a function $f(t)$ that is expressed in the form of an infinite series (Chap. 6) whose terms contain the above trigonometric functions multiplied by arbitrary constant coefficients:

$$f(t)=a_0+(a_1\cos\omega t+b_1\sin\omega t)+(a_2\cos 2\omega t+b_2\sin 2\omega t)+\ldots+$$
$$+(a_n\cos n\omega t+b_n\sin n\omega t)+\ldots$$

or

$$f(t)=\sum_{n=0}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t) \qquad (2)$$

The function $f(t)$ is periodic with period $T=2\pi/\omega$; that is, $f(t+T)=f(t)$.

Note: It can be proven that *every* periodic function with period T can be expanded into a series of the form (2), with $\omega=2\pi/T$ and suitable coefficients a_n, b_n. This series is called *Fourier series* [1,2].

Exercise 1.7 Show that a function $f(t)$ expressed in the Fourier-series form

$$f(t)=\sum_{n=0}^{\infty}\left(a_n\cos\frac{2\pi n t}{T}+b_n\sin\frac{2\pi n t}{T}\right)$$

is periodic with period T.

Exercise 1.8 Examine the periodicity (or not) of the following functions:

(1) $f(x) = \sin 2x + \cos \sqrt{2}\, x$ (2) $f(x) = 2\cos(x/2) - 5\sin(x/3)$
(3) $f(x) = 5\sin 2x - 3\cos^2 x$ (4) $f(x) = \tan \lambda x$

1.10 Inverse Function

Let $y=f(x)$ be a function and let D be its domain of definition. The *range* of f is the set $B=\{ f(x) / x \in D\} \equiv f(D)$. The function defines a *mapping* of the set D onto the set B, such that to every point $x \in D$ there corresponds a *unique* point $y \in B$. If, moreover, to every point $y \in B$ there corresponds a *unique* point $x \in D$, the mapping is called *bijective* or "one-to-one" (1-1). In this case,

$$x_1 = x_2 \;\Leftrightarrow\; f(x_1) = f(x_2) \quad \text{or, equivalently,} \quad x_1 \neq x_2 \;\Leftrightarrow\; f(x_1) \neq f(x_2) .$$

A function $y=f(x)$ which is 1-1 is called *invertible* since it allows us to define the *inverse function* $x= f^{-1}(y)$, with domain of definition B and range D, so that

$$f^{-1}[f(x)] = x \;, \qquad f[f^{-1}(y)] = y .$$

We notice that

$$(f^{-1} \circ f)(x) = x \;, \qquad (f \circ f^{-1})(y) = y .$$

We say that the composition of f and f^{-1} is the *identity function*.

Examples:

1. The function $y= f(x)= x^3$ is 1-1, with $D=B=R$. The inverse function is $x = f^{-1}(y) = \sqrt[3]{y}$.

2. The function $y= f(x)= e^x$ is 1-1, with $D=R$ and $B=R^+$. The inverse function is $x= f^{-1}(y) = \ln y$.

3. The function $y= f(x)= x^2$, with $D=R$ and $B= [0, +\infty)$, is *not* 1-1 since to every value $y>0$ there correspond *two* values $x = \pm\sqrt{y}$. Thus this function is *not* invertible (the inverse function is *multiple-valued*; see Sec. 1.4).

4. The function $y= f(x)=\sin x$, with $D=R$ and $B= [-1, 1]$, is not 1-1 since to every value $y \in [-1, 1]$ there correspond *infinitely many* values of $x= \arcsin y$. Thus this function is not invertible.

1.11 Monotonicity of a Function

Consider a function $y= f(x)$ with domain of definition D, and let $[a, b] \subseteq D$ be an interval on the x-axis. The function f is said to be *increasing* in $[a, b]$ if, for any $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$, while f is *decreasing* in the considered interval if, for any $x_1, x_2 \in [a,b]$ with $x_1 < x_2$, we have $f(x_1) > f(x_2)$. A function that is either increasing or decreasing in some interval is said to be *monotone* in that interval.

Exercise 1.9 Show that a function f that is monotone in its *entire* domain of definition is *invertible*, and the inverse function f^{-1} also is monotone (increasing or decreasing, in accordance with f).

Examples:

1. The linear function $y= ax+b$ is increasing for $a>0$ and decreasing for $a<0$.

2. The function $y = e^x$ is increasing on the entire x-axis.

3. The function $y = e^{-x}$ is decreasing on the entire x-axis.

4. The function $y = x^2$ is decreasing in $(-\infty, 0]$ and increasing in $[0, +\infty)$.

Exercise 1.10 Verify the above statements.

References

1. A. F. Bermant, I. G. Aramanovich, *Mathematical Analysis* (Mir Publishers, 1975).
2. M. D. Greenberg, *Advanced Engineering Mathematics*, 2nd Edition (Prentice-Hall, 1998).

CHAPTER 2

DERIVATIVE AND DIFFERENTIAL

2.1 Definition

In a sense, the derivative is a "measure of sensitivity" of a function $y= f(x)$ to small changes of *x*. The larger the change of *y*, the bigger is the sensitivity of the function. This sensitivity generally depends on *x*. This observation leads to the definition of a new function $y' = f'(x)$, called the *derivative* of $f(x)$.

Let $y= f(x)$ be a continuous function. We consider an arbitrary change of *x*, namely, $x \to x+\Delta x$. This induces a corresponding change of *y*: $y \to y+\Delta y$, where

$$\Delta y = f(x+\Delta x) - f(x) \quad \text{so that} \quad y+\Delta y = f(x) + \Delta y = f(x+\Delta x) \ .$$

A measure of the sensitivity of $f(x)$ at point *x* is the quotient $\Delta y/\Delta x$, provided that Δx is very small. We have:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \ .$$

This expression is a function of *two* independent variables, namely, *x* and Δx. If, however, we take the limit of $\Delta y/\Delta x$ for $\Delta x \to 0$, the result will depend only on *x*, i.e., it will be a function of *x*. This function is called the *derivative* of $f(x)$ and is denoted $f'(x)$:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \ .$$

We will often write: $y' = f'(x)$. The value of $f'(x)$ at a particular point $x=x_0$ is

$$f'(x_0) \equiv f'(x)\,|_{x=x_0} \ .$$

If this value exists, the function is said to be *differentiable* at x_0. The process of finding the derivative of a function is called *differentiation* (to *differentiate* a function means to find its derivative).

Examples:

1. $y= f(x)= c$ (constant function). We have:

$$\Delta y = f(x+\Delta x) - f(x) = c - c = 0 \ \Rightarrow \ \frac{\Delta y}{\Delta x} = 0 \ \Rightarrow \ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0 \ .$$

Thus,

$$y' = (c)' = 0 \ .$$

2. $y=f(x)=ax+b$ (linear function). We have:

$$\Delta y = f(x+\Delta x)-f(x) = [a(x+\Delta x)+b]-(ax+b) = a\Delta x \;\Rightarrow$$
$$\frac{\Delta y}{\Delta x} = a \;\Rightarrow\; \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = a$$

Thus,

$$y' = (ax+b)' = a \;.$$

3. $y=f(x)=x^2$. We have:

$$\Delta y = f(x+\Delta x)-f(x) = (x+\Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2 \;\Rightarrow$$
$$\frac{\Delta y}{\Delta x} = 2x+\Delta x \;\Rightarrow\; \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = 2x$$

Thus,

$$y' = (x^2)' = 2x \;.$$

4. $y=f(x)=x^3$. We have:

$$\Delta y = f(x+\Delta x)-f(x) = (x+\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \;\Rightarrow$$
$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2 \;\Rightarrow\; \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = 3x^2$$

Thus,

$$y' = (x^3)' = 3x^2 \;.$$

5. $y=f(x)=x^a$ $(a\in R)$. As can be proven [1,2],

$$y' = (x^a)' = a\,x^{a-1} \;.$$

Let us see some examples:

$$\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -x^{-2} = -\frac{1}{x^2}$$
$$\left(\frac{1}{x^2}\right)' = \left(x^{-2}\right)' = -2x^{-3} = -\frac{2}{x^3}$$
$$\left(\sqrt{x}\right)' = \left(x^{1/2}\right)' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$
$$\left(\frac{1}{\sqrt{x}}\right)' = \left(x^{-1/2}\right)' = -\frac{1}{2}x^{-3/2} = -\frac{1}{2\sqrt{x^3}}$$

The derivative of a function admits a geometrical interpretation to be discussed in Sec. 2.10.

2.2 Differentiation Rules

1. Derivative of a sum or difference of functions:

$$\left(f_1(x) \pm f_2(x) \pm \cdots \right)' = f_1'(x) \pm f_2'(x) \pm \cdots$$

- *The derivative of a sum or difference of functions equals the sum or difference, respectively, of the derivatives of these functions.*

2. Derivative of a product of functions (*Leibniz rule*):

$$\left(f_1(x) f_2(x) \right)' = f_1'(x) f_2(x) + f_1(x) f_2'(x)$$
$$\left(f_1(x) f_2(x) f_3(x) \right)' = f_1'(x) f_2(x) f_3(x) + f_1(x) f_2'(x) f_3(x) + f_1(x) f_2(x) f_3'(x)$$

etc. In particular, if c is a constant, then $(c)'=0$ and

$$[\, c f(x)]' = c f'(x) \,.$$

3. Derivative of a quotient of functions:

$$\left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Exercise 2.1 Find the derivatives of the following functions:

(1) $y = \dfrac{\sqrt{x}}{2} - \dfrac{2}{\sqrt{x}}$ (2) $y = \dfrac{\sin x}{x}$ (3) $y = 2\sqrt{x^3}\, e^x - \dfrac{3 \ln x}{x}$

The following important theorem will be proven in the Appendix:

If the derivative of a function $f(x)$ is defined at a point $x=x_0$, the function is continuous at x_0.

It should be noted carefully that the converse of this theorem is *not* true, in general! Indeed, a function may be continuous at a point where its derivative is not defined. For example, the direction of the graph of $f(x)$ may change abruptly at some point $x=x_0$, as seen in Fig. 2.1. The derivative $f'(x)$ will then be non-continuous at x_0, in accordance with the geometrical interpretation of the derivative (to be discussed in Sec. 2.10).

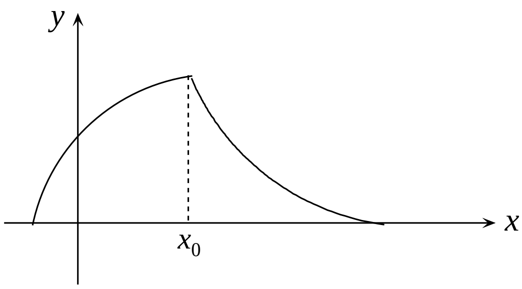


Fig. 2.1. A continuous function with a non-continuous derivative at x=x_0 .

2.3 Derivatives of Trigonometric Functions

1. For the function $y=f(x)=\sin x$ we have:

$$f'(x)=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x} \quad .$$

But,

$$\sin(x+\Delta x)-\sin x=2\sin\frac{(x+\Delta x)-x}{2}\cos\frac{(x+\Delta x)+x}{2}=2\sin\frac{\Delta x}{2}\cos\frac{2x+\Delta x}{2} \quad .$$

So,

$$f'(x)=\lim_{\Delta x\to 0}\frac{2\sin\frac{\Delta x}{2}\cos\frac{2x+\Delta x}{2}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}\cos\frac{2x+\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$=\left(\lim_{\frac{\Delta x}{2}\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right)\left(\lim_{\Delta x\to 0}\cos\frac{2x+\Delta x}{2}\right)=1\cdot\cos\frac{2x+0}{2}$$

where we have used the fact that $\lim_{u\to 0}\frac{\sin u}{u}=1$. Thus,

$$\boxed{(\sin x)'=\cos x}$$

2. For $y=f(x)=\cos x$,

$$f'(x)=\lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x} \quad .$$

We have:

$$\cos(x+\Delta x)-\cos x=2\sin\frac{(x+\Delta x)+x}{2}\sin\frac{x-(x+\Delta x)}{2}=-2\sin\frac{\Delta x}{2}\sin\frac{2x+\Delta x}{2} \quad .$$

Thus,

$$f'(x)=\lim_{\Delta x\to 0}\frac{-2\sin\frac{\Delta x}{2}\sin\frac{2x+\Delta x}{2}}{\Delta x}=-\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}\sin\frac{2x+\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$=-\left(\lim_{\frac{\Delta x}{2}\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right)\left(\lim_{\Delta x\to 0}\sin\frac{2x+\Delta x}{2}\right)=-1\cdot\sin\frac{2x+0}{2}$$

and therefore

$$\boxed{(\cos x)'=-\sin x}$$

3. For the function $y=f(x)=\tan x$ we have:

$$f'(x)=\left(\frac{\sin x}{\cos x}\right)'=\frac{(\sin x)'\cos x-\sin x\,(\cos x)'}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}\ \Rightarrow$$

$$\boxed{(\tan x)'=\frac{1}{\cos^2 x}}$$

Similarly,

$$\boxed{(\cot x)'=-\frac{1}{\sin^2 x}}$$

2.4 Table of Derivatives of Elementary Functions

$(c)'=0\quad(c=const.)$	$(\sin x)'=\cos x$	$(\arcsin x)'=\frac{1}{\sqrt{1-x^2}}$
$(x^\alpha)'=\alpha x^{\alpha-1}\quad(\alpha\in R)$	$(\cos x)'=-\sin x$	$(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}$
$(e^x)'=e^x$	$(\tan x)'=\frac{1}{\cos^2 x}$	$(\arctan x)'=\frac{1}{1+x^2}$
$(\ln x)'=\frac{1}{x}$	$(\cot x)'=-\frac{1}{\sin^2 x}$	$(\mathrm{arc}\cot x)'=-\frac{1}{1+x^2}$

2.5 Derivatives of Composite Functions

Let $y=f(u)$ and $u=\varphi(x)$ be two differentiable functions. We define the composite function

$$y = (f \circ \varphi)(x) \equiv f[\varphi(x)] \ .$$

The derivative of this function with respect to x is equal to

$$y' = (f \circ \varphi)'(x) = f'(u)\,\varphi'(x) \ .$$

Proof: We write: $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u}\dfrac{\Delta u}{\Delta x}$. Since φ is continuous (why?), $\Delta u \to 0$ when $\Delta x \to 0$. Now,

$$y' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}\right) = \left(\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\right) = f'(u)\varphi'(x) \ .$$

We will adopt the simpler notation $y=y(u)$ and $u=u(x)$ so that, by composition of these functions, $y=y(x)$. We thus write

$$y'(x) = y'(u)\, u'(x) \ .$$

Similarly, by composition of $y=y(u)$, $u=u(w)$ and $w=w(x)$ we have $y=y(x)$ and

$$y'(x) = y'(u)\, u'(w)\, w'(x) \ ,$$

etc. The above differentiation rule for composite functions is often called the "chain rule".

Examples:

1. $y(x)=e^{2x}$. We write $y(u)=e^u$, $u(x)=2x$. Then,

$$y'(x) = y'(u)u'(x) = (e^u)'(2x)' = 2e^u \ \Rightarrow \ (e^{2x})' = 2e^{2x} \ .$$

2. $y(x)=e^{-x}$. We write $y(u)=e^u$, $u(x)=-x$. Then,

$$y'(x) = y'(u)u'(x) = (e^u)'(-x)' = -e^u \ \Rightarrow \ (e^{-x})' = -e^{-x} \ .$$

3. In general,

$$\boxed{(e^{ax})' = ae^{ax} \ \ (a \in R)}$$

4. $y(x)=e^{\sqrt{x^2+1}}$. Write $y(u)=e^u$, $u(w)=\sqrt{w}=w^{1/2}$, $w(x)=x^2+1$. Then,

$$y'(x)=y'(u)\,u'(w)\,w'(x)=(e^u)'(w^{1/2})'(x^2+1)'=e^u\left(\frac{1}{2}w^{-1/2}\right)(2x+0)=\frac{xe^u}{\sqrt{w}}=\frac{xe^{\sqrt{w}}}{\sqrt{w}}\Rightarrow$$

$$\left(e^{\sqrt{x^2+1}}\right)'=\frac{x}{\sqrt{x^2+1}}\,e^{\sqrt{x^2+1}}\ .$$

5. In general,

$$\boxed{\left(e^{f(x)}\right)'=f'(x)\,e^{f(x)}}$$

6. As can be easily shown,

$$\boxed{(\sin ax)'=a\cos ax\ ,\quad (\cos ax)'=-a\sin ax\quad (a\in R)}$$

More generally,

$$\boxed{\begin{array}{ll}[\sin f(x)]'=f'(x)\cos f(x)\ , & [\cos f(x)]'=-f'(x)\sin f(x)\\ [\tan f(x)]'=f'(x)/\cos^2 f(x)\ , & [\cot f(x)]'=-f'(x)/\sin^2 f(x)\end{array}}$$

7. $y(x)=\ln(\sin x)$. Write $y(u)=\ln u$, $u(x)=\sin x$. Then,

$$y'(x)=y'(u)\,u'(x)=(\ln u)'(\sin x)'=\frac{1}{u}\cos x=\frac{\cos x}{\sin x}\Rightarrow$$

$$\boxed{[\ln(\sin x)]'=\cot x}$$

Similarly,

$$\boxed{[\ln(\cos x)]'=-\tan x}$$

More generally,

$$\boxed{[\ln f(x)]'=\frac{f'(x)}{f(x)}}$$

8. In general,

$$\boxed{\left([f(x)]^a\right)'=a\,[f(x)]^{a-1}f'(x)\quad (a\in R)}$$

For example,

$$(\sin^2 x)'\equiv[(\sin x)^2]'=2\sin x(\sin x)'=2\sin x\cos x=\sin 2x$$

$$(\sqrt{\ln x})'\equiv[(\ln x)^{1/2}]'=\frac{1}{2}(\ln x)^{-1/2}(\ln x)'=\frac{1}{2x\sqrt{\ln x}}$$

2.6 Derivatives of Functions of the Form $y=[f(x)]^{\varphi(x)}$

Consider a function of the form $y = [f(x)]^{\varphi(x)}$, where $f(x)>0$ for all values of x in some subset of the domain of definition of f. We want to find the derivative y' with respect to x.

Technique: We write

$$f(x)=e^{\ln f(x)} \quad \text{so that} \quad y=\left[e^{\ln f(x)}\right]^{\varphi(x)}=e^{\varphi(x)\ln f(x)}\equiv e^{g(x)}.$$

Then,

$$y'=[e^{g(x)}]'=g'(x)\,e^{g(x)}=g'(x)\,e^{\varphi(x)\ln f(x)}=[\varphi(x)\ln f(x)]'\,[f(x)]^{\varphi(x)}$$
$$=[\varphi(x)\ln f(x)]'\,y$$

Examples:

1. $y=a^x \ (a>0)$. We write

$$a=e^{\ln a} \quad\Rightarrow\quad y=a^x=(e^{\ln a})^x=e^{x\ln a}$$
$$y'=(e^{x\ln a})'=(x\ln a)'\,e^{x\ln a}=(\ln a)\,a^x$$

That is,

$$\boxed{(a^x)'=(\ln a)\,a^x \quad (a>0)}$$

2. $y=x^x \ (x>0)$. We write

$$x=e^{\ln x} \quad\Rightarrow\quad y=x^x=(e^{\ln x})^x=e^{x\ln x}$$
$$y'=(e^{x\ln x})'=(x\ln x)'\,e^{x\ln x}=(1+\ln x)\,x^x$$

That is,

$$\boxed{(x^x)'=(1+\ln x)\,x^x \quad (x>0)}$$

Exercise 2.2 Find the derivatives of the following functions:

(1) $y=e^{\sqrt[3]{\sin(3x^2+1)}}$ (2) $y=\cos^2\left(\sqrt[3]{x^6+1}\right)$ (3) $y=\tan(\sin^3 2x)$

(4) $y=\ln[\ln(x^4+1)]$ (5) $y=\sqrt{\ln\sqrt{x^2+1}}$ (6) $y=(x+1)^{x+1} \ (x>-1)$

(7) $y=x^{x^2} \ (x>0)$ (8) $y=(\sin x)^{\cos x} \ (0<x<\pi)$ (9) $y=\ln|x| \ (x\neq 0)$

[Hint for (9): $|x|=x$ if $x>0$ while $|x|=-x$ if $x<0$. Examine the two cases separately. What do you observe?]

2.7 Differential of a Function

Consider a function $y = f(x)$. Let Δx be an arbitrary change of the independent variable, from an initial value x to $x+\Delta x$. The corresponding change of y is

$$\Delta y = f(x+\Delta x) - f(x) \ .$$

Note that Δy is a function of *two* independent variables, x and Δx.

The derivative of f at a point x has been defined as

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} \ .$$

This suggests that a function $\varepsilon(x,\Delta x)$ must exist such that

$$\frac{\Delta y}{\Delta x} = f'(x) + \varepsilon(x,\Delta x) \quad \text{where} \quad \lim_{\Delta x\to 0} \varepsilon(x,\Delta x) = 0 \ .$$

Thus,

$$\Delta y = f'(x)\,\Delta x + \varepsilon(x,\Delta x)\Delta x \tag{1}$$

The product $f'(x)\Delta x$ is *linear* (i.e., of the first degree) in Δx, while the product $\varepsilon(x,\Delta x)\Delta x$ must only contain terms *of the second degree and higher* in Δx (that is, it may not contain a constant term as well as a linear term). We write, symbolically,

$$\varepsilon(x,\Delta x)\Delta x \equiv O(\Delta x^2) \quad \text{where} \quad \Delta x^2 \equiv (\Delta x)^2 \ \left(\neq \Delta(x^2)\,!\right) \ .$$

Equation (1) is then written

$$\boxed{\Delta y = f'(x)\,\Delta x + O(\Delta x^2)} \tag{2}$$

We observe that Δy is the sum of a linear and a higher-order term in Δx. Furthermore, the derivative of f at x is the coefficient of Δx in the linear term.

Example: Let $y = f(x) = x^3$. Then,

$$\Delta y = f(x+\Delta x) - f(x) = (x+\Delta x)^3 - x^3 = 3x^2\,\Delta x + (3x\Delta x^2 + \Delta x^3)$$

from which we have that $f'(x) = 3x^2$ and $O(\Delta x^2) = 3x\Delta x^2 + \Delta x^3$.

The linear term in (2), which is a function of x and Δx, is called the *differential* of the function $y = f(x)$ and is denoted dy:

$$\boxed{dy = d\,f(x) = f'(x)\,\Delta x} \tag{3}$$

Equation (2) is then written

$$\Delta y = dy + O(\Delta x^2) \tag{4}$$

If Δx is *infinitesimal* ($|\Delta x| << 1$) we can make the approximation $O(\Delta x^2) \approx 0$. Hence,

$$\Delta y \approx dy = f'(x)\,\Delta x \quad \textit{for infinitesimal } \Delta x\,.$$

Note however that, for *finite Δx* the *difference Δy* and the *differential dy* are separate quantities, in general!

An exception is the case of linear functions. Let $y = f(x) = ax+b$. Then,

$$\Delta y = f(x+\Delta x) - f(x) = [a(x+\Delta x)+b] - (ax+b) = a\Delta x$$

and

$$dy = f'(x)\,\Delta x = (ax+b)'\Delta x = a\Delta x = \Delta y\,.$$

That is, *for linear functions (and only for these functions) the differential dy is the same as the difference Δy*, even if these quantities assume finite values. This means that, for linear functions, $O(\Delta x^2) = 0$.

Let us see a few applications of the definition (3) of the differential:

$$\text{For } f(x) = x^a \;\Rightarrow\; d(x^a) = (x^a)'\Delta x = a\,x^{a-1}\Delta x$$
$$\text{For } f(x) = e^x \;\Rightarrow\; d(e^x) = (e^x)'\Delta x = e^x\Delta x$$
$$\text{For } f(x) = \ln x \;\Rightarrow\; d(\ln x) = (\ln x)'\Delta x = \frac{1}{x}\Delta x$$

For the function $f(x) = x$ we have: $dx = (x)'\Delta x = 1\cdot\Delta x \;\Rightarrow$

$$\boxed{\Delta x = dx}$$

in accordance with our earlier remark regarding linear functions. Relation (3) may thus be rewritten more symmetrically as

$$\boxed{dy = d\,f(x) = f'(x)\,dx}$$

Dividing this by *dx*, we obtain the following expression for the derivative:

$$\boxed{f'(x) = \frac{dy}{dx} = \frac{d\,f(x)}{dx}}$$

In words: *The derivative of a function is equal to the differential of the function divided by the differential (or, equivalently, the change) of the independent variable.*

Exercise 2.3 Verify the following properties of the differential:

1. $d\,[\,f(x) \pm g(x)] = df(x) \pm dg(x)$
2. $d\,[\,f(x)\,g(x)] = f(x)\,dg(x) + g(x)\,df(x)$
3. $d\,[c\,f(x)] = c\,df(x) \quad (c=const.)$
4. $d\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)df(x) - f(x)dg(x)}{[g(x)]^2}$

2.8 Differential Operators

We introduce a notation that proves to be important in higher mathematics:

$$\frac{d\,f(x)}{dx} \equiv \frac{d}{dx}\,f(x)\ .$$

Notice that this notation attempts to "mimic" the properties of ordinary multiplication of numbers:

$$\frac{\alpha \cdot \beta}{\gamma} = \frac{\alpha}{\gamma} \cdot \beta$$

except that the expression $\dfrac{d}{dx}$ is definitely *not* a number! The symbol $\dfrac{d}{dx}$ is called a *differential operator* and, when placed in front of a function $f(x)$, it *instructs* us to take the derivative of $f(x)$. We thus write:

$$\boxed{f'(x) = \frac{d\,f(x)}{dx} = \frac{d}{dx}\,f(x)}$$

The above relation exhibits three different notations for the derivative of a function!

Note the following properties:

1. $\dfrac{d}{dx}[\,f(x) \pm g(x)] = \dfrac{df(x)}{dx} \pm \dfrac{dg(x)}{dx} = \dfrac{d}{dx}\,f(x) \pm \dfrac{d}{dx}\,g(x)$
2. $\dfrac{d}{dx}[\,f(x)\,g(x)] = \dfrac{df(x)}{dx}\,g(x) + f(x)\dfrac{dg(x)}{dx} = f(x)\dfrac{d}{dx}\,g(x) + g(x)\dfrac{d}{dx}\,f(x)$

In words: The differential operator is a *linear* operator that satisfies the *Leibniz rule* (Sec. 2.2). Operators having these properties are called *derivations* and are of great importance in physical theories such as electrodynamics and quantum mechanics.

2.9 Derivative of a Composite Function by Using the Differential

We consider two functions f and g such that $y=f(u)$ and $u=g(x)$. As we know, the composite function $(f \circ g)$ is defined by the relation

$$y = (f \circ g)(x) \equiv f\,[\,g(x)\,]\,.$$

To simplify our notation, we write $y=y(u)$, $u=u(x)$ and $y=y(x)=y\,[u(x)]$.

We want to find an expression for the derivative of y with respect to x. This derivative is equal to the quotient dy/dx. We write:

$$y'(x) = \frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = y'(u)\,u'(x)$$

which expresses the familiar "chain rule" for the derivative of a composite function (see Sec. 2.5).

2.10 Geometrical Significance of the Derivative and the Differential

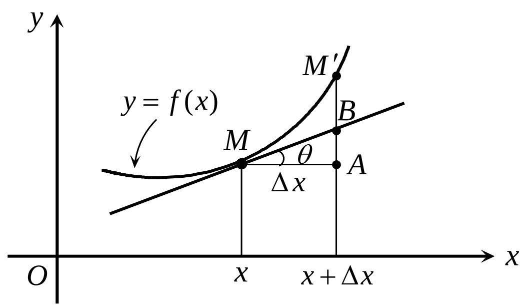


Fig. 2.2. Graph of a function $y=f(x)$ and the tangent line at point M.

Figure 2.2 shows a section of the graph of a function $y=f(x)$. We consider an arbitrary point $M \equiv (x, y)$ of the curve and we draw the tangent line to this curve at M. This line forms an angle θ with the x-axis. As we see in the figure, to the change $\Delta x=MA$ of x there corresponds the change $\Delta y=AM'$ of y. The linear section AB then represents the differential dy of f for the given values of x and Δx, while the derivative of f at x is equal to $\tan\theta$. Indeed,

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{MA \to 0} \frac{AM'}{MA} = \lim_{BM' \to 0} \frac{AM'}{MA} = \frac{AB}{MA} = \tan\theta$$

where we have used the fact that $BM' \to 0$ when $MA \to 0$. Therefore,

- *the value of the derivative of the function $y=f(x)$ for some given x is equal to the slope of the line tangent to the graph of $f(x)$ at the point $M \equiv (x, y)$*

(cf. Sec. 1.6). We also have:

$$dy = f\,'(x)\,\Delta x = (\tan\theta)\,\Delta x = \frac{AB}{MA}\,MA = AB\,.$$

Finally, from equation (4) of Sec. 2.7 we have that

$$O(\Delta x^2) = \Delta y - dy = AM' - AB = BM'\,.$$

If the function f is *linear*, then $B \equiv M'$ so that $O(\Delta x^2) = 0$ and $\Delta y = dy = AB$.

2.11 Higher-Order Derivatives

The *second derivative* of a function $y = f(x)$ is defined as follows:

$$f\,''(x) \equiv [f\,'(x)]' = \frac{d}{dx}\frac{df(x)}{dx} = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \left(\frac{d}{dx}\right)^2 f(x) = \frac{d^2}{dx^2}f(x)$$

or

$$y'' = f\,''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d^2 y}{dx^2}$$

where $dx^2 \equiv (dx)^2$. In an analogous way we define the *third derivative*:

$$y''' = f\,'''(x) \equiv [f\,''(x)]' = \frac{d^3}{dx^3}f(x) = \frac{d^3 f(x)}{dx^3} = \frac{d^3 y}{dx^3}\quad .$$

In general, the nth-*order derivative* of $y = f(x)$ is written:

$$y^{(n)} = f^{(n)}(x) = \frac{d^n}{dx^n}f(x) = \frac{d^n f(x)}{dx^n} = \frac{d^n y}{dx^n}\quad .$$

Examples:

1. $(x^a)' = a\,x^{a-1}$, $(x^a)'' = a(a-1)x^{a-2}$, $(x^a)''' = a(a-1)(a-2)x^{a-3}$, $\cdots$ $(a \in R)$

2. $(\sin x)' = \cos x$, $(\sin x)'' = -\sin x$, $(\sin x)''' = -\cos x$, $(\sin x)'''' = \sin x$, etc.

3. $(e^x)' = (e^x)'' = (e^x)''' = \cdots = e^x$

Note, in particular, that the simple exponential function $y=e^x$ is the *only* function that is equal to its derivative (and, therefore, to its derivatives of all orders). In fact, it is by this property that the function $y=e^x$ is often defined.

Exercise 2.4 For any two functions $u(x)$ and $v(x)$, show the following:

1. $(u+v)'' = u'' + v''$, $(u+v)''' = u''' + v'''$, etc.

2. $(uv)'' = u''v + 2u'v' + uv''$
   $(uv)''' = u'''v + 3u''v' + 3u'v'' + uv'''$

2.12 Derivatives of Implicit Functions

Let the algebraic relation $F(x, y)=0$ define an implicit function (Sec. 1.4). In principle, by this relation the variable y may be regarded as a function of x: $y=y(x)$. There is, however, no simple mathematical formula that would *explicitly* express y in terms of x. How then will we find the derivative $y'(x)$?

In this case we work as follows: we differentiate the relation $F(x, y)=0$ with respect to x, keeping in mind that y is *implicitly* a function of x.

Examples:

1. Let $F(x, y) \equiv x^2+y^2-1=0$ (unit circle on the xy-plane). Taking the x-derivative,

$$\frac{d}{dx}(x^2+y^2-1)=0 \;\Rightarrow\; 2x+\frac{d(y^2)}{dy}\frac{dy}{dx}=0 \;\Rightarrow\; 2x+2yy'=0 \;\Rightarrow\; y'=-\frac{x}{y} \quad .$$

2. Let $F(x, y) \equiv y^3-3xy+x^3=0$. Taking the x-derivative, we find:

$$3y^2y'-3y-3xy'+3x^2=0 \;\Rightarrow\; y'=\frac{y-x^2}{y^2-x} \quad .$$

3. Let $F(x, y) \equiv e^y - x = 0$ ($x>0$), which is equivalent to $e^y = x$ or $y=\ln x$. Taking the derivative of $F(x, y)=0$ with respect to x, we find the familiar expression for the derivative of the logarithmic function:

$$y'e^y-1=0 \;\Rightarrow\; y'=e^{-y}=\frac{1}{x} \quad .$$

References

1. D. D. Berkey, *Calculus*, 2nd Edition (Saunders College, 1988).
2. A. F. Bermant, I. G. Aramanovich, *Mathematical Analysis* (Mir Publishers, 1975).

CHAPTER 3

SOME APPLICATIONS OF DERIVATIVES

3.1 Tangent and Normal Lines on Curves

Consider a function $y = f(x)$ and let $M \equiv (x_0, y_0)$ [where $y_0 = f(x_0)$] be a point of its graph on the xy-plane (Fig. 3.1). We call $y = T(x)$ the *linear* function describing the *tangent line* to the curve $f(x)$ at point M, and we call $y = N(x)$ the equation of the line *normal* to the tangent line at M. The lines $T(x)$ and $N(x)$ are, therefore, perpendicular to each other. We seek the explicit equations describing these lines.

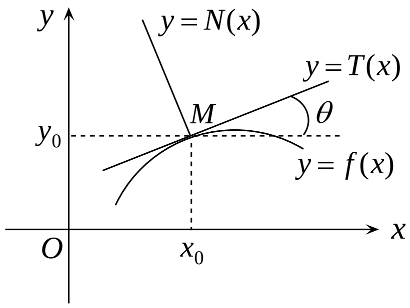


Fig. 3.1. Tangent line $y=T(x)$ and normal line $y=N(x)$ to the curve $y=f(x)$.

Tangent line $\boldsymbol{y = T(x)}$

As we saw in Sec. 1.6, a line passing through (x_0, y_0) and having slope $a = \tan\theta$ is described mathematically by the equation

$$y - y_0 = a(x - x_0) .$$

Also, according to Sec. 2.10 the slope of the tangent to the curve $y = f(x)$ at (x_0, y_0) is equal to $a = f'(x_0)$. Hence the equation of the tangent line is

$$\boxed{y - y_0 = (x - x_0) f'(x_0)}$$

Normal line $\boldsymbol{y = N(x)}$

This line passes through (x_0, y_0) and forms an angle $(\theta + \pi/2)$ with the x-axis; thus its slope is $a' = \tan(\theta + \pi/2) = -\cot\theta = -1/\tan\theta = -1/a$, where $a = \tan\theta = f'(x_0)$ is the slope of the tangent line. The equation of the normal line is, therefore,

$$y - y_0 = a'(x - x_0) = -(1/a)(x - x_0) \quad \Rightarrow$$

$$\boxed{y - y_0 = -(x - x_0) / f'(x_0)}$$

3.2 Angle of Intersection of Two Curves

We consider two curves C_1 and C_2 described, respectively, by the functions $y=f_1(x)$ and $y=f_2(x)$. The curves intersect at a point $M\equiv(x_0, y_0)$, where $f_1(x_0)=f_2(x_0)=y_0$ (Fig. 3.2). Let $y=T_1(x)$ and $y=T_2(x)$ be the lines tangent to C_1 and C_2 at M. We seek the angle φ formed by these two tangents.

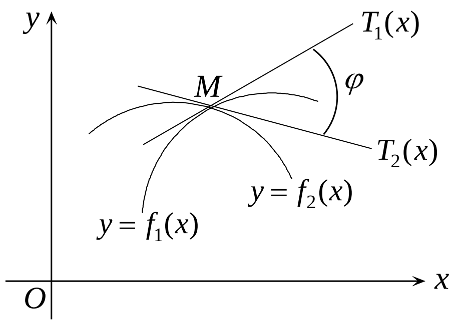


Fig. 3.2. Angle φ of intersection of the curves $y=f_1(x)$ and $y=f_2(x)$ at point M.

Let θ_1 and θ_2 be the angles formed by the two tangents with the x-axis (we assume that $\theta_1>\theta_2$). The angle between these tangents is then $\varphi=\theta_1-\theta_2$. Now, the slopes of the two lines are equal to

$$a_1=\tan\theta_1=f_1'(x_0)\,, \qquad a_2=\tan\theta_2=f_2'(x_0)\,.$$

Therefore,

$$\tan\varphi=\tan(\theta_1-\theta_2)=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2} \;\Rightarrow$$

$$\boxed{\tan\varphi=\frac{a_1-a_2}{1+a_1a_2}=\frac{f_1'(x_0)-f_2'(x_0)}{1+f_1'(x_0)\,f_2'(x_0)}}$$

Special cases:

1. If $a_1=a_2$ then $\tan\varphi=0$ and $\varphi=0$. That is, the two tangent lines coincide.

2. If $a_1=-1/a_2 \Leftrightarrow 1+a_1a_2=0$, then $\tan\varphi=\infty$ and $\varphi=\pi/2$. That is, the two tangents intersect at right angles.

Note: Consider, in general, two lines on the xy-plane, having slopes $a_1=\tan\theta_1$ and $a_2=\tan\theta_2$. The angle $\varphi=\theta_1-\theta_2$ formed by these lines is then given by the relation

$$\boxed{\tan\varphi=\frac{a_1-a_2}{1+a_1a_2}}$$

In particular, if $a_1=a_2$ the two lines are *parallel* to each other, while if $a_1=-1/a_2 \Leftrightarrow 1+a_1a_2=0$ the lines are *perpendicular* to each other.

Examples:

1. Let $y= f(x)= e^{2x}$. We seek the equations of the tangent and the normal line at the point $(x_0, y_0) \equiv (0,1)$. We have: $f'(x_0)= f'(0)= 2$. Thus, for the tangent line,

$$y - y_0 = (x - x_0)\, f'(x_0) \;\Rightarrow\; y - 1 = (x - 0)\, f'(0) \;\Rightarrow\; y = 2x+1$$

while for the normal line,

$$y - y_0 = -(x - x_0) / f'(x_0) \;\Rightarrow\; y - 1 = -(x - 0)/2 \;\Rightarrow\; y = -x/2 + 1 \,.$$

We notice that the slopes of the two lines are, respectively, $a=2$ and $a' = -1/2$, so that the condition of perpendicularity, $1+ a\, a' = 0$, is satisfied.

2. Consider the lines $y= f_1(x)= x$ and $y= f_2(x)= -x$. We call (x_0, y_0) their point of intersection. At that point, $f_1(x_0)= f_2(x_0)= y_0$. Obviously, $x_0 = y_0 = 0 \Leftrightarrow (x_0, y_0) \equiv (0,0)$. Now, the slopes of these lines are $a_1=1$ and $a_2= -1$. We observe that $1+ a_1 a_2 = 0$, which means that the two lines intersect at right angles at $(0,0)$.

3.3 Maximum and Minimum Values of a Function

Consider a function $y=f(x)$. We say that $f(x)$ is *increasing* at $x=x_0$ if for $h>0$, sufficiently small,

$$f(x_0-h) < f(x_0) < f(x_0+h) \,.$$

Similarly, $f(x)$ is *decreasing* at $x=x_0$ if

$$f(x_0-h) > f(x_0) > f(x_0+h) \,.$$

The following can be proven:

- If $f'(x_0) > 0$ then $f(x)$ is *increasing* at $x=x_0$.
- If $f'(x_0) < 0$ then $f(x)$ is *decreasing* at $x=x_0$.
- If $f'(x_0) = 0$ then $f(x)$ is *stationary* at $x=x_0$.

A point (x_0, y_0) at which $f'(x_0)= 0$ is called a *critical point* of $y=f(x)$.

The function $y=f(x)$ has a *local maximum* at $x=x_0$ if for $h>0$, sufficiently small,

$$f(x_0) > f(x_0-h) \quad\text{and}\quad f(x_0) > f(x_0+h) \,,$$

while it has a *local minimum* at $x=x_0$ if

$$f(x_0) < f(x_0-h) \quad\text{and}\quad f(x_0) < f(x_0+h)$$

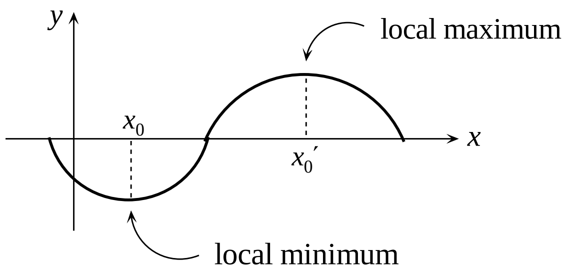


Fig. 3.3. A local maximum and a local minimum of a function.

(see Fig. 3.3). In general, a (local) maximum or minimum of $f(x)$ is called an *extremum* (extreme value) of this function.

There are two methods for determining the maxima and minima of a function:

First-derivative test

1. We solve the equation $f\,'(x)=0$ to find the critical points of $y=f(x)$.

2. Let $x=x_0$ be a critical point and let $h>0$, sufficiently small. Then,

- $f(x_0)$ is a *maximum* if $f\,'(x_0-h)>0$ and $f\,'(x_0+h)<0$;
- $f(x_0)$ is a *minimum* if $f\,'(x_0-h)<0$ and $f\,'(x_0+h)>0$;
- $f(x_0)$ is *neither a maximum nor a minimum* if $f\,'(x_0-h)\,f\,'(x_0+h)\geq 0$.

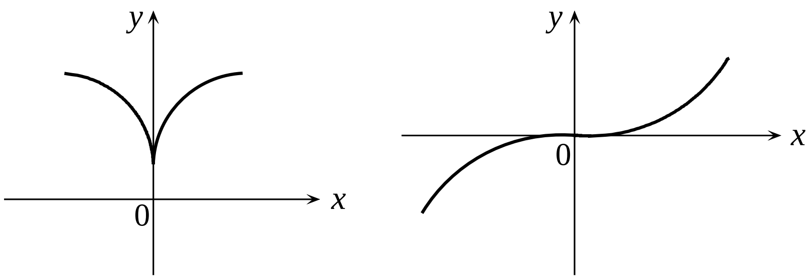
Second-derivative test

1. We solve the equation $f\,'(x)=0$ to find the critical points of $y=f(x)$.

2. Let $x=x_0$ be a critical point. Then,

- if $f\,''(x_0)<0$, $f(x_0)$ is a *maximum* ;
- if $f\,''(x_0)>0$, $f(x_0)$ is a *minimum* ;
- if $f\,''(x_0)=0$ or ∞ , the test fails (we use the first-derivative test instead).

Comment: The condition $f\,'(x_0)=0$ is *neither necessary nor sufficient* in order that the critical point $x=x_0$ be an extremum of $y=f(x)$! This is demonstrated in Fig. 3.4.



(α) (β)

Fig. 3.4. In case (α) the function has a minimum at x=0, although its derivative does not vanish there. In case (β) we have $f\,'(0)=0$ but the point x=0 is not an extremum (it is neither a maximum nor a minimum).

Exercise 3.1 Study the functions $y=\sin x$ and $y=\cos x$. Find (a) the critical points, (b) the intervals where each function is increasing or decreasing, and (c) the maximum and minimum values of y in each case.

3.4 Indeterminate Forms and L'Hospital's Rule

The process of finding the limit of a function for $x \to x_0$ often leads to expressions that cannot be defined mathematically. The most common types of such *indeterminate forms* are the following:

$$\frac{0}{0}\,, \quad \frac{\infty}{\infty}\,, \quad 0\cdot\infty\,, \quad \infty-\infty\,, \quad 0^0\,, \quad 1^\infty, \quad \infty^0$$

Problems of this kind are treated by using *L'Hospital's theorem* [1,2].

Theorem: Let $f(x)$ and $g(x)$ be two functions such that

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0 \quad \text{or} \quad \lim_{x\to x_0} f(x) = \pm \lim_{x\to x_0} g(x) = \infty$$

(where x_0 may be finite or infinite). Then,

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f\,'(x)}{g\,'(x)} \quad .$$

If $\lim_{x\to x_0} f\,'(x) = \lim_{x\to x_0} g\,'(x) = 0$ or ∞, then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f\,''(x)}{g\,''(x)} \quad \text{(and so forth)}\ .$$

By this theorem we treat the cases $0/0$ and ∞/∞ directly.

The case $0\cdot\infty$ reduces to the previous ones as follows: Assume that $f(x)\to 0$ and $g(x)\to\infty$. We then write

$$f(x)\cdot g(x)=\frac{f(x)}{1/g(x)}\to\frac{0}{0}\quad\text{or}\quad f(x)\cdot g(x)=\frac{g(x)}{1/f(x)}\to\frac{\infty}{\infty}\ .$$

The case $\infty-\infty$ is treated as follows: Let $f(x)\to+\infty$ and $g(x)\to+\infty$. We write

$$f(x)-g(x)=\frac{1}{1/f(x)}-\frac{1}{1/g(x)}=\frac{(1/g)-(1/f)}{1/(f\cdot g)}\to\frac{0}{0}\ .$$

The cases 0^0, $1^{+\infty}$ and $(+\infty)^0$ are treated by using the transformation

$$[f(x)]^{g(x)}=\left[e^{\ln f(x)}\right]^{g(x)}=e^{g(x)\cdot\ln f(x)}$$

and by taking into account that $\lim\limits_{x\to x_0} e^{h(x)}=\exp\left[\lim\limits_{x\to x_0} h(x)\right]$.

Examples:

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}\ (0/0)=\lim\limits_{x\to 0}\dfrac{\cos x}{1}=1$.

2. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x-\sin x}\ (0/0)=\lim\limits_{x\to 0}\dfrac{\sin x}{1-\cos x}\ (0/0)=\lim\limits_{x\to 0}\dfrac{\cos x}{\sin x}=\infty$.

3. For $a>0$, $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x^a}\ (\infty/\infty)=\lim\limits_{x\to+\infty}\dfrac{1}{a x^a}=0$ (we say that x^a tends to infinity *faster than* $\ln x$).

4. For $n>0$, $\lim\limits_{x\to 0^+}(x^n\ln x)\ (0\cdot\infty)=\lim\limits_{x\to 0^+}\dfrac{\ln x}{1/x^n}\ (\infty/\infty)=-\lim\limits_{x\to 0^+}\dfrac{x^n}{n}=0$.

5. $\lim\limits_{x\to 1^+}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)\ (\infty-\infty)=\lim\limits_{x\to 1^+}\dfrac{x\ln x-x+1}{(x-1)\ln x}\ (0/0)$

$$=\lim_{x\to 1^+}\frac{\ln x}{\ln x+\dfrac{x-1}{x}}\ (0/0)=\lim_{x\to 1^+}\frac{1/x}{(1/x)+(1/x^2)}=\frac{1}{2}\ .$$

6. Let $A=\lim\limits_{x\to 0^+}x^x\ (0^0)$. We write: $A=\lim\limits_{x\to 0^+}e^{x\ln x}=\exp\left[\lim\limits_{x\to 0^+}(x\ln x)\right]$.
According to Example 4, $\lim\limits_{x\to 0^+}(x\ln x)=0$. Thus, $A=1$. Symbolically we write $0^0=1$, in the sense that $\lim\limits_{x\to 0^+}x^x=1$.

Exercise 3.2 Find the following limits:

(1) $\lim\limits_{x\to 0}\dfrac{x - x\cos x}{x - \sin x}$

(2) $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x} - \cot x\right)$

(3) $\lim\limits_{x\to 0}(\cos x)^{1/x^2}$

(4) $\lim\limits_{x\to 0^+}(\cot x)^{1/\ln x}$

References

1. D. D. Berkey, *Calculus*, 2nd Edition (Saunders College, 1988).
2. A. F. Bermant, I. G. Aramanovich, *Mathematical Analysis* (Mir Publishers, 1975).

CHAPTER 4

INDEFINITE INTEGRAL

4.1 Antiderivatives of a Function

Definition: Consider a function $f(x)$. Every function $F(x)$ whose derivative is equal to $F'(x) = f(x)$ constitutes an *antiderivative* of $f(x)$.

If $F(x)$ is an antiderivative of $f(x)$ then *every* function $G(x)=F(x)+C$, where C is any constant, also is an antiderivative of $f(x)$ (show this!). Thus, given a function $f(x)$ and an antiderivative $F(x)$ of $f(x)$ we can find an *infinite* set of antiderivatives of $f(x)$; namely, $\{F(x)+C \,/\, C\in R\}$.

The infinite set $I=\{F(x)+C \,/\, C\in R\}$, where $F(x)$ is *any* antiderivative of $f(x)$, contains *all* antiderivatives of $f(x)$; that is, there are no antiderivatives of $f(x)$ that do not belong to the set I. This conclusion is based on the following theorem:

Theorem: Any two antiderivatives of a function $f(x)$ can differ at most by a constant.

Proof: Let $F(x)$ and $G(x)$ be two antiderivatives of $f(x)$. Then,

$$F'(x) = G'(x) = f(x) \Leftrightarrow F'(x) - G'(x) \equiv [F(x) - G(x)]' = 0 \Leftrightarrow F(x) - G(x) = C \,.$$

According to this theorem, the set $I=\{F(x)+C \,/\, C\in R\}$ of antiderivatives of $f(x)$ is uniquely defined, regardless of the choice of the particular antiderivative $F(x)$. Indeed, any other antiderivative $G(x)$ will differ from $F(x)$ only by a constant and, therefore, $G(x)$ itself will belong to the set I. In conclusion:

- To find the (infinite) set of *all* antiderivatives of $f(x)$ it suffices to find *any* antiderivative $F(x)$ and construct the set $I = \{F(x)+C \,/\, C\in R\}$ for all values of the real constant C.

Symbol: Omitting the brackets (*which, however, will <u>always</u> be assumed to exist!*) we will denote the *set I* of antiderivatives of $f(x)$ as follows:

$$I = F(x) + C \quad (\text{all } C\in R) \,.$$

Examples:

1. The set of antiderivatives of $f(x)=x^2$ is $I= x^3/3 + C$.

2. The set of antiderivatives of $f(x)=e^{2x}$ is $I= e^{2x}/2 + C$.

3. The set of antiderivatives of $f(x)=-2/x \; (x>0)$ is $I= -2\ln x + C$.

4.2 The Indefinite Integral

Definition: The infinite set I of all antiderivatives of a function $f(x)$ is called the *indefinite integral* of this function and is denoted $I=\int f(x)dx$. The function $f(x)$ is called the *integrand*, while x is called the *variable of integration*.

If $F(x)$ is *any* antiderivative of $f(x)$: $F'(x) = f(x)$, then the indefinite integral I is given by the expression

$$\boxed{I=\int f(x)\,dx=F(x)+C}$$

for all real values of the constant C. We emphasize again that I represents an *infinite set* of functions, not any particular function! If we insisted on being notationally accurate, we should write $I=\int f(x)dx=\{F(x)+C\,/\,C\in R\}$. Thus, strange as it may seem, the following relation is true:

$$\int f(x)\,dx=\int f(x)\,dx+C'\ ,\quad \forall C'\in R \quad (!)$$

(imagine that we add C' to all elements of I). This, of course, expresses equality between *sets*, not between particular functions. Given that $F'(x) = f(x)$, we may write

$$\boxed{\int F'(x)\,dx=F(x)+C} \tag{1}$$

for any function $F(x)$.

The symbol dx inside the integral sign is called the *"differential"*. It should not be perceived, however, as an actual differential in the way it was defined in Chap. 2, nor should it be interpreted as an infinitesimal quantity! To understand the spirit of this notation, let us temporarily change the symbol dx to δx and write (1) as

$$\int F'(x)\,\delta x=F(x)+C \tag{2}$$

For $F(x)=x$ this yields $\int\delta x=x+C$. Putting u in place of x, $\int\delta u=u+C$. Now, let us suppose that u is a function of x: $u=f(x)$. Then, $\int\delta f(x)=f(x)+C$. On the other hand, according to (2) we have $\int f'(x)\delta x=f(x)+C$. We notice that $\int\delta f(x)=\int f'(x)\delta x$, which allows us to write, symbolically, $\delta f(x)=f'(x)\delta x$. This, of course, *resembles* the definition of the differential: $df(x) = f'(x)dx$! Moreover, it is not hard to prove that the symbols δ and d share common properties when placed in front of functions. We thus call δx the "differential" of integration and write relation (2) in the form (1). We also write:

$$\int dF(x)=\int F'(x)\,dx=F(x)+C\ .$$

Basic Table of Integrals

$$\int dx = x + C$$

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C \qquad (a \neq -1)$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int \cos x\, dx = \sin x + C$$

$$\int \sin x\, dx = -\cos x + C$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int \frac{dx}{1+x^2} = \arctan x + C$$

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

$$\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln\left(x + \sqrt{x^2 \pm 1}\right) + C$$

4.3 Basic Integration Rules

1. Integral of a sum or difference of functions:

$$\int [f(x) \pm g(x) \pm \cdots]\, dx = \int f(x)\, dx \pm \int g(x)\, dx \pm \cdots$$

How are we to interpret the "sum of sets" on the right-hand side? Let $F(x)$ and $G(x)$ be any antiderivatives of $f(x)$ and $g(x)$, respectively. Then, by definition,

$$\int f(x)\, dx + \int g(x)\, dx \equiv \{F(x) + G(x) + C \,/\, C \in R\} \equiv F(x) + G(x) + C \;.$$

2. Any constant multiplicative factor may be taken outside the integral:

$$\int c\, f(x)\, dx = c \int f(x)\, dx \qquad (c{=}const.)$$

Combining the above two properties, we have:

$$\int [c_1\, f(x) \pm c_2 g(x) \pm \cdots]\, dx = c_1 \int f(x)\, dx \;\pm\; c_2 \int g(x)\, dx \pm \cdots$$

3. As we have already mentioned,

$$\boxed{\int d\, f(x) = \int f\,'(x)\, dx = f(x) + C}$$

4. Change of variable of integration:

Assume that $\int f(x)dx{=}F(x){+}C$, where $F(x)$ is an antiderivative of $f(x)$: $F\,'(x){=}f(x)$. Renaming the variable x to u, we write: $\int f(u)du{=}F(u){+}C$, where $F\,'(u){=}f(u)$. Now, suppose that the variable u is a function of x: $u{=}u(x)$. Then,

$$\boxed{\int f(u)\, du = \int f\big(u(x)\big)\, u'(x)\, dx = F\big(u(x)\big) + C}$$

This property plays an important role in the method of integration by substitution, to be studied in the next section.

Exercise 4.1 Compute the following integrals:

(1) $\displaystyle\int \left(\frac{2}{x^2} - \frac{3}{x} + \frac{1}{2\sqrt{x}} \right) dx$

(2) $\displaystyle\int \left(3 - \frac{2}{x} + 4\sqrt{x} \right) dx$

4.4 Integration by Substitution (Change of Variable)

Assume that we are given the integral $I=\int f(x)dx$, where $f(x)$ is not an elementary function. It is often possible to find a new variable u, which is a function of x: $u=u(x)$, such that the integral I takes on the form $I=\int g(u)du$, where $g(u)$ is now an elementary (or, at any rate, simpler) function. If

$$\int g(u)du = F(u) + C \quad ,$$

then

$$I = F\,[u(x)] + C \quad .$$

Notice that

$$I = \int g(u)\,du = \int g[u(x)]\,u'(x)\,dx = \int f(x)dx$$

which means that our aim is to set the given function $f(x)$ in the form

$$f(x) = g[u(x)]\,u'(x)$$

and then let $u'(x)$ be "absorbed" into the differential dx, thus creating a new differential du.

As an example, let $f(x)$ be of the form $f(x)=u'(x)/u(x)$, so that $g(u)=1/u$. Then, assuming that $u(x)>0$,

$$I = \int \frac{u'(x)}{u(x)}dx = \int \frac{du}{u} = \ln\big(u(x)\big) + C \quad .$$

Some useful transformations of the differential

$$dx = d\,(x+c)$$

$$dx = \frac{1}{a}\,d\,(ax) \quad (a \neq 0)$$

$$x^a dx = \frac{1}{a+1}\,d\left(x^{a+1}\right) \quad (a \neq -1)$$

$$x^{-1}dx = \frac{dx}{x} = d\,(\ln x)$$

$$e^{ax}dx = \frac{1}{a}\,d\left(e^{ax}\right) \quad (a \neq 0)$$

$$\cos ax\,dx = \frac{1}{a}\,d\,(\sin ax) \quad , \quad \sin ax\,dx = -\frac{1}{a}\,d\,(\cos ax) \quad (a \neq 0)$$

Exercise 4.2 Verify the above relations.

Examples:

1. $I = \int x e^{x^2} dx$.

We write $xdx = \frac{1}{2} d(x^2)$ and we set $u=x^2$. Then,

$$I = \frac{1}{2}\int e^{x^2} d(x^2) = \frac{1}{2}\int e^u du = \frac{1}{2}(e^u + C') = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C \quad (\text{where } C = C'/2) .$$

2. $I = \int \frac{x^2 dx}{2x^3+1}$.

We write $x^2 dx = \frac{1}{3}d(x^3) = \frac{1}{6}d(2x^3) = \frac{1}{6}d(2x^3+1)$ and we set $u=2x^3+1$:

$$I = \frac{1}{6}\int \frac{d(2x^3+1)}{2x^3+1} = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}(\ln u + C') = \frac{1}{6}\ln u + C = \frac{1}{6}\ln(2x^3+1) + C \quad (C=C'/6) .$$

3. $I = \int \frac{\ln x}{x} dx$.

We write $\frac{1}{x}dx = d(\ln x)$ and we set $u=\ln x$:

$$I = \int \ln x \, d(\ln x) = \int u \, du = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C \; .$$

4. $I = \int \frac{x\ln(x^2+1)}{x^2+1} dx$.

By writing $xdx = \frac{1}{2}d(x^2) = \frac{1}{2}d(x^2+1)$ and by setting $u=x^2+1$, we have:

$I = \frac{1}{2}\int \frac{\ln u}{u} du$, which is of the form of Example 3 (with u in place of x).

By making the new substitution $w=\ln u$, show that $I = \frac{1}{4}[\ln(x^2+1)]^2 + C$.

5. $I = \int \tan x \, dx \quad (0 < x < \pi/2)$.

We write:

$$I = \int \frac{\sin x}{\cos x} dx = -\int \frac{d(\cos x)}{\cos x} \quad (\text{set } u = \cos x) \; = -\int \frac{du}{u} = -(\ln u + C') \;\Rightarrow$$

$$\boxed{\int \tan x \, dx = -\ln(\cos x) + C}$$

(where $C = -C'$). Similarly, we find:

$$\boxed{\int \cot x \, dx = \ln(\sin x) + C}$$

6. $I = \int \frac{\tan\sqrt{x}}{\sqrt{x}}\, dx$.

By writing $\frac{1}{\sqrt{x}}\, dx = x^{-1/2}dx = \frac{1}{1/2}\, d(x^{1/2}) = 2\, d(\sqrt{x})$ and by setting $u = \sqrt{x}$, we find: I=2 $\int \tan u\, du$, which takes us back to Example 5. The result is

$$I = -2\ln(\cos\sqrt{x}) + C \ .$$

7. $I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$.

By writing $(e^x - e^{-x})\, dx = d\,(e^x + e^{-x})$ and by setting $u = e^x + e^{-x}$, we have:

$$I = \int \frac{du}{u} = \ln u + C = \ln(e^x + e^{-x}) + C \ .$$

8. $I = \int \frac{\sin 2x}{1+\sin^2 x}\, dx$.

This is written:

$$I = 2\int \frac{\sin x \cos x}{1+\sin^2 x}\, dx = 2\int \frac{\sin x\, d(\sin x)}{1+\sin^2 x} \quad (\text{set } u = \sin x) \quad = 2\int \frac{u\, du}{1+u^2}$$
$$= 2\ \frac{1}{2}\int \frac{d(u^2)}{1+u^2} = \int \frac{d(1+u^2)}{1+u^2} \quad (\text{set } w = 1+u^2) \quad = \int \frac{dw}{w} = \ln w + C$$
$$= \ln(1+u^2) + C = \ln(1+\sin^2 x) + C \ .$$

Exercise 4.3 Verify the results in the above examples by showing that the derivative of each expression found equals the function that was to be integrated.

Exercise 4.4 Find the following integrals and verify your results:

(1) $\int \frac{dx}{x\ln x}$ $(x > 1)$ (2) $\int \frac{e^{1/x^2}}{x^3}\, dx$ (3) $\int \frac{x\ln(x^2+1)}{x^2+1}\, dx$

(4) $\int \frac{dx}{\tan x\, \cos^2 x}$ $(0 < x < \pi/2)$ (5) $\int \frac{e^{\sqrt{x}}\cos(e^{\sqrt{x}})}{\sqrt{x}}\, dx$ (6) $\int \frac{dx}{x^2+5}$

(7) $\int \frac{dx}{x^2 - 6x + 18}$ (*Hint:* Write the denominator as a sum of squares)

4.5 Integration by Parts (Partial Integration)

The method of partial integration is used for integrals of the form

$$I = \int u(x)\, v'(x)\, dx = \int u(x)\, dv(x)$$

when the method of substitution (change of variable) is not applicable.

Theorem: Consider the functions $u=u(x)$ and $v=v(x)$. The following equality of *sets* is true:

$$\int u(x)\, v'(x)\, dx \;=\; u(x)\, v(x) - \int v(x)\, u'(x)\, dx \quad \Leftrightarrow$$

$$\int u\, dv = uv - \int v\, du$$

(imagine that the product uv is added to every element of the infinite set on the right-hand side).

Proof: As mentioned in Sec. 4.2, the "differential" inside the integral sign shares common properties with the ordinary differential of functions. Thus,

$$d(uv) = u\, dv + v\, du \;\Rightarrow\; u\, dv = d(uv) - v\, du \;\Rightarrow\; \int u\, dv = \int d(uv) - \int v\, du \quad \Rightarrow$$

$$\int u\, dv = (uv + C) - \int v\, du = uv - \left(\int v\, du - C\right) = uv - \int v\, du \;,$$

given that the infinite *sets* $\int v\, du$ and $\left(\int v\, du - C\right)$ coincide.

Method: Suppose we are given an integral of the form $I=\int f(x)\, g(x)\, dx$, which cannot be computed by the method of substitution. We seek an antiderivative $h(x)$ of $g(x)$ and we write

$$I = \int f(x)\, h'(x)\, dx = \int f(x)\, d\, h(x) = f(x)\, h(x) - \int h(x)\, d\, f(x)$$

$$= f(x)\, h(x) - \int h(x)\, f'(x)\, dx \,.$$

If this transformation does not lead to a simpler integration relative to the initial one, we seek an antiderivative of $f(x)$ and we work in a similar way. In certain cases, two successive partial integrations yield an algebraic equation for I that is easy to solve.

Examples:

1. $I = \int x\, e^x dx$.

If we choose to put x inside the differential and then apply partial integration, we will end up with an even harder integral containing x^2 in place of x! We thus try putting the exponential factor inside the differential:

$$I = \int x\, d(e^x) = xe^x - \int e^x dx = x\, e^x - (e^x + C') = (x-1)\, e^x + C \;.$$

2. $I = \int \ln x \, dx$.

We have: $I = x\ln x - \int x \, d(\ln x) = x\ln x - \int x \frac{1}{x} dx = x\ln x - (x + C')$ $\Rightarrow$

$$\boxed{\int \ln x \, dx = x(\ln x - 1) + C}$$

3. $I = \int x \ln x \, dx$.

We put x inside the differential: $I = \frac{1}{2}\int \ln x \, d(x^2)$ $\Rightarrow$

$2I = \int \ln x \, d(x^2) = x^2 \ln x - \int x^2 d(\ln x) = x^2 \ln x - \int x \, dx = x^2 \ln x - (\frac{x^2}{2} + C')$ $\Rightarrow$

$I = \frac{x^2}{2}(\ln x - \frac{1}{2}) + C$.

4. $I = \int x^2 \cos x \, dx$.

We put the trigonometric function inside the differential:

$I = \int x^2 d(\sin x) = x^2 \sin x - 2\int x \sin x \, dx = x^2 \sin x - 2 I_1$, where

$I_1 = \int x \sin x \, dx = -\int x \, d(\cos x) = -x\cos x + \int \cos x \, dx = -x\cos x + \sin x + C'$. Hence,

$I = (x^2 - 2)\sin x + 2x\cos x + C$.

5. $I = \int e^x \cos x \, dx$.

We put the exponential function inside the differential:

$I = \int \cos x \, d(e^x) = e^x \cos x - \int e^x d(\cos x) = e^x \cos x + I_1$, where

$I_1 = \int e^x \sin x \, dx = \int \sin x \, d(e^x) = e^x \sin x - \int e^x d(\sin x) = e^x \sin x - \int e^x \cos x \, dx$
$= e^x \sin x - I$.

Thus, $I = e^x \cos x + e^x \sin x - I$ $\Rightarrow$ $2I = e^x(\cos x + \sin x) + C'$ $\Rightarrow$
$I = \frac{e^x}{2}(\cos x + \sin x) + C$.

Comment: Why was it necessary to add the constant C' in the expression for $2I$? (Remember that I is a *set*!)

Exercise 4.5 Find the following integrals:

(1) $\int x^2 e^x \, dx$ (2) $\int e^x \sin x \, dx$

(3) $\int \sin^2 x \, dx$ (*without* making a trigonometric transformation of $\sin^2 x$!)

(4) $\int \cos^2 x \, dx$ (similarly)

Some integration problems are composite. Specifically, a change of variable transforms the given integral to a form that is integrable by parts.

Examples:

1. $I = \int x^5 e^{x^3} dx$.

We write $I = \int x^3 e^{x^3} x^2 dx = \frac{1}{3} \int x^3 e^{x^3} d(x^3)$ (set $u = x^3$) $= \frac{1}{3} \int u \, e^u du$,

which can be integrated by parts. We find:

$I = \frac{1}{3} (x^3 - 1) \, e^{x^3} + C$.

2. $I = \int e^{\sqrt{x}} dx$.

We write $I = \int \sqrt{x} \, e^{\sqrt{x}} \frac{1}{\sqrt{x}} dx = 2 \int \sqrt{x} \, e^{\sqrt{x}} d(\sqrt{x})$ (set $u = \sqrt{x}$) $= 2 \int u \, e^u du$,

which is integrable by parts. We find:

$I = 2 (\sqrt{x} - 1) e^{\sqrt{x}} + C$.

Exercise 4.6 Compute the following integrals:

(1) $\int \frac{e^{\sqrt{x}} \cos \sqrt{x}}{\sqrt{x}} \, dx$

(2) $\int \sin 2x \, \ln(\sin x) \, dx$ $(0 < x < \pi / 2)$

4.6 Integration of Rational Functions

A *proper rational function* is a fractional function of the form $R(x)=P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials and where the degree of $P(x)$ is *less* than that of $Q(x)$. (In general, any rational fraction can be written as $P(x)/Q(x)=S(x)+P_1(x)/Q(x)$, where $S(x)$ and $P_1(x)$ are polynomials and the degree of $P_1(x)$ is less than that of $Q(x)$. We will only consider *proper* rational functions here.)

Let us assume that $\deg[Q(x)]=n$ (where "deg" means "degree"). Without loss of generality, the coefficient of the highest-order term x^n in $Q(x)$ is taken to be 1. That is,

$$Q(x) \equiv x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0 .$$

If $\rho_1, \rho_2, \cdots, \rho_n$ are the roots of $Q(x)$ (not necessarily all different) then

$$Q(x) \equiv (x-\rho_1)(x-\rho_2)\cdots(x-\rho_n) .$$

If some root, say ρ_1, is complex, then its complex conjugate will also be a root (call it $\rho_2 = \overline{\rho_1}$). Thus, given that $x \in R$,

$$(x-\rho_1)(x-\rho_2) = (x-\rho_1)(\overline{x-\rho_1}) \equiv x^2 + px + q$$

where $p^2 - 4q < 0$. If the complex root ρ_1 is of multiplicity l, then $Q(x)$ will contain the factor $(x-\rho_1)^l (x-\rho_2)^l = (x^2+px+q)^l$. Thus, finally, $Q(x)$ will be of the form

$$Q(x) \equiv (x-a)^k \cdots (x^2+px+q)^l \cdots \qquad (a \in R,\quad p^2 - 4q < 0)$$

where a is a real root of multiplicity k and where the equation $x^2 + px + q = 0$ has complex conjugate roots.

Theorem: The rational function $R(x)=P(x)/Q(x)$, where $\deg[P(x)] < \deg[Q(x)]$, can be decomposed into a sum of *partial fractions*, as follows:

$$\frac{P(x)}{Q(x)} \equiv \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k} + \cdots + \frac{B_1 x + C_1}{x^2+px+q} + $$
$$+ \frac{B_2 x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_l x + C_l}{(x^2+px+q)^l} + \cdots$$

where A_i, B_i, C_i are constants to be determined.

Example: Let $Q(x) \equiv (x^2-4)(x+1)^2(x^2+1)^2$. We write

$$Q(x) \equiv (x-2)(x+2)(x+1)^2(x^2+1)^2 .$$

Assume that $\deg[P(x)] < 8$. Then,

$$\frac{P(x)}{Q(x)} \equiv \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} .$$

Method: Suppose we are given an integral of the form $I=\int R(x)dx$, where $R(x)=P(x)/Q(x)$ is a proper rational function. We decompose $R(x)$ into a sum of partial fractions of the forms $A/(x-a)^k$ and $(Bx+C)/(x^2+px+q)^l$. Hence the integral I becomes a sum of integrals of the forms $\int dx/(x-a)^k$, $\int dx/(x^2+px+q)^l$ and $\int xdx/(x^2+px+q)^l$.

Examples:

1. $I=\int \dfrac{x-5}{x^3-3x^2+4}\,dx \quad (x>2)$. [*Hint:* $x^3-3x^2+4=(x+1)(x-2)^2$]

We write $\dfrac{x-5}{x^3-3x^2+4}=\dfrac{x-5}{(x+1)(x-2)^2}\equiv\dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$.

The constant coefficients satisfy the equations

$A+B=0$, $C-4A-B=1$, $4A-2B+C=-5 \;\Rightarrow\; A=-2/3$, $B=2/3$, $C=-1$. Thus

$$I=-\frac{2}{3}\int\frac{dx}{x+1}+\frac{2}{3}\int\frac{dx}{x-2}-\int\frac{dx}{(x-2)^2} \;=\; \frac{2}{3}\ln\left(\frac{x-2}{x+1}\right)+\frac{1}{x-2}+C \ .$$

2. $I=\int \dfrac{x+1}{x^3-x^2+x-1}\,dx \quad (x>1)$. [*Hint:* $x^3-x^2+x-1=(x-1)(x^2+1)$]

We write $\dfrac{x+1}{x^3-x^2+x-1}=\dfrac{x+1}{(x-1)(x^2+1)}\equiv\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$.

The constant coefficients satisfy the equations

$A+B=0$, $C-B=1$, $A-C=1 \;\Rightarrow\; A=1$, $B=-1$, $C=0$. Thus

$$I=\int\frac{dx}{x-1}-\int\frac{xdx}{x^2+1} \;=\; \ln\left(\frac{x-1}{\sqrt{x^2+1}}\right)+C \ .$$

CHAPTER 5

DEFINITE INTEGRAL

5.1 Definition and Properties

Let $f(x)$ be a function and let $F(x)$ be any one of its antiderivatives: $F'(x)=f(x)$. As we know, the *infinite set* of all antiderivatives of $f(x)$ is represented by the *indefinite integral*

$$\int f(x)\,dx = F(x) + C \quad (C\in R) .$$

Now, let a, b be real constants. We define the *definite integral* of $f(x)$ from a to b as the real *number*

$$\boxed{\int_a^b f(x)\,dx \equiv F(b)-F(a) \equiv [F(x)]_a^b}$$

The constants a and b are called the *limits* (*lower* and *upper*, respectively) of integration.

Examples:

1. $\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = \sin(\pi/2)-\sin 0 = 1$.

Similarly,

$$\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = -\cos(\pi/2)+\cos 0 = 1 .$$

But,

$$\int_0^{\pi/2} \cos 2x\,dx = [\frac{1}{2}\sin 2x]_0^{\pi/2} = \frac{1}{2}\sin\pi - \frac{1}{2}\sin 0 = 0 .$$

2. $\int_a^b \frac{dx}{x} = [\ln x]_a^b = \ln b - \ln a = \ln(b/a) \quad (a>0,\ \ b>0)$.

Exercise 5.1 For $0< a<\pi/2$ and $0<b<\pi/2$, show that

(1) $\int_a^b \cot x\,dx = \ln\left(\frac{\sin b}{\sin a}\right)$ (2) $\int_a^b \tan x\,dx = \ln\left(\frac{\cos a}{\cos b}\right)$

Properties of the definite integral

1. The value of the integral is independent of the choice of the antiderivative $F(x)$ of $f(x)$. Indeed, if $G(x)=F(x)+C$ is any other antiderivative, then

$$[G(x)]_a^b = G(b)-G(a) = \big(F(b)+C\big)-\big(F(a)+C\big) = F(b)-F(a) = [F(x)]_a^b \ .$$

2. The value of the integral is independent of the name of the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du = \cdots$$

For example, $\int_0^a x^2\,dx = \int_0^a u^2\,du = [x^3/3]_0^a = [u^3/3]_0^a = a^3/3$.

3. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

4. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx \quad (c = const.)$

5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$, $\qquad \int_a^a f(x)\,dx = 0$

6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (c \in R)$ (show this!)

7. $\int_a^b d\,f(x) = \int_a^b f'(x)\,dx = f(b)-f(a)$ (*Newton-Leibniz formula*)

5.2 Integration by Substitution

Consider the definite integral $I = \int_{x_1}^{x_2} f(x)\,dx$. Assume that we can find a transformation of the form $u=\varphi(x)$, such that the indefinite integration with respect to u is easier to perform relative to that with respect to x. Specifically, assume that

$$\int f(x)\,dx = \int g(u)\,du = G(u) + C \tag{1}$$

where $G(u)$ is an antiderivative of $g(u)$. We can work in two ways:

1. We first find the indefinite integral $\int f(x)dx$ by making the substitution $u=\varphi(x)$. According to (1),

$$\int f(x)\,dx = G[\varphi(x)] + C \equiv F(x) + C$$

where $F(x)$ is an antiderivative of $f(x)$. Then the definite integral I will be equal to

$$I=\int_{x_1}^{x_2} f(x)\,dx = [F(x)]_{x_1}^{x_2} = F(x_2)-F(x_1)\ .$$

2. We transform the *definite* integral *I* directly into one with respect to *u*, taking into account that *a change of variable u=φ(x) implies a corresponding change of the limits of integration*:

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{u_1}^{u_2} g(u)\,du \;=\; [G(u)]_{u_1}^{u_2} = G(u_2)-G(u_1)$$
$$\textit{where}\quad u_1=\varphi(x_1)\ ,\ \ u_2=\varphi(x_2)\ .$$

Examples:

1. $I=\int_0^2 x e^{x^2} dx$.

Let us first find the corresponding indefinite integral:

$$\int x e^{x^2} dx = \frac{1}{2}\int e^{x^2} d(x^2)\ \ (\text{set } u=x^2)\ = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C\ .$$

Then, $I=\frac{1}{2}[e^{x^2}]_0^2 = \frac{1}{2}(e^4-1)$.

Alternatively, we evaluate the definite integral directly:

$$I=\int_0^2 x e^{x^2} dx = \frac{1}{2}\int_0^2 e^{x^2} d(x^2)\ .$$

We set $u=x^2$ and transform the integral with respect to x into an integral for u, *not forgetting to adjust the limits of integration also*:

$\int_0^2 dx \to \int_0^4 du$. Thus, $I=\frac{1}{2}\int_0^4 e^u du = \frac{1}{2}[e^u]_0^4 = \frac{1}{2}(e^4-1)$.

2. $I=\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$.

We write $I=-\int_0^{\pi/2}\frac{d(\cos x)}{1+\cos^2 x}$. We make the transformation

$u=\cos x$, $\int_0^{\pi/2} dx \to \int_1^0 du$. Then,

$$I=-\int_1^0 \frac{du}{1+u^2} = \int_0^1 \frac{du}{1+u^2} = [\arctan u]_0^1 = \frac{\pi}{4}-0\ \Rightarrow$$

$$\boxed{\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{4}}$$

3. $I = \int_0^2 \frac{x\ln(x^2+1)}{x^2+1}\,dx$.

We write $I = \frac{1}{2}\int_0^2 \frac{\ln(x^2+1)}{x^2+1}\,d(x^2+1)$ and we make the substitution

$u = x^2+1$, $\int_0^2 dx \to \int_1^5 du$. Then, $I = \frac{1}{2}\int_1^5 \frac{\ln u}{u}\,du = \frac{1}{2}\int_1^5 \ln u\, d(\ln u)$.

We set $w = \ln u$, $\int_1^5 du \to \int_0^{\ln 5} dw$. Then, $I = \frac{1}{2}\int_0^{\ln 5} w\,dw = \frac{1}{4}(\ln 5)^2$.

Exercise 5.2 Compute the following integrals:

(1) $\int_0^2 \frac{x}{1+x^2}\,dx$ (2) $\int_0^{\pi/6} \cos x\, e^{\sin x}\,dx$ (3) $\int_1^e \frac{\sin\left(\frac{\pi}{2}\ln x\right)}{x}\,dx$

5.3 Integration of Even, Odd and Periodic Functions

Consider an integral of the form

$$I = \int_{-a}^{a} f(x)\,dx$$

over some "symmetric" interval of integration [$-a$, a] (we assume a>0). We write

$$I = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \equiv I_1 + I_2 \quad .$$

The integral I_1 is written $I_1 = \int_{-a}^{0} f(x)\,dx = -\int_{-a}^{0} f\left(-(-x)\right) d(-x)$.

We perform the transformation $u = -x$, $\int_{-a}^{0} dx \to \int_a^0 du$:

$$I_1 = -\int_a^0 f(-u)\,du = \int_0^a f(-u)\,du \quad .$$

As we have mentioned, the value of a definite integral does not change if we give a different name to the integration variable. Hence we may now put x in place of u :

$$I_1 = \int_0^a f(-x)\,dx \quad .$$

Finally, $I = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \;\Rightarrow$

$$\boxed{\int_{-a}^{a} f(x)\,dx = \int_0^a [\,f(-x) + f(x)]\,dx}$$

This relation is valid for *any* function $f(x)$ defined in the interval $[-a\,,a]$. Such a function, of course, need be neither even nor odd. If, however, it belongs to one of these categories, then, as we know (Sec. 1.8),

$$\begin{aligned} f(-x) + f(x) &= 2\,f(x) \quad \text{if } f(x) \text{ is } \textit{even}\,, \\ &= \;\; 0 \qquad\quad \text{if } f(x) \text{ is } \textit{odd}\,. \end{aligned}$$

Thus,

$$\boxed{\begin{aligned} \int_{-a}^{a} f(x)\,dx &= 2\int_0^a f(x)\,dx \quad \text{if } f(x) \text{ is } \textit{even} \\ &= \;0 \qquad\qquad\qquad\quad \text{if } f(x) \text{ is } \textit{odd} \end{aligned}}$$

Exercise 5.3 Justify the following results by inspection (i.e., without performing any integration):

(1) $\int_{-a}^{a} \sin(kx)\,dx = 0 \quad (a\,,\,k \in R)$

(2) $\int_{-a}^{a} \cos(kx)\,dx = 2\int_0^a \cos(kx)\,dx$

(3) $\int_{-\pi/3}^{\pi/3} x^3 \tan x \sin(x^5 - 2x^3 + 6x)\,dx = 0$

(4) $\int_{-1}^{1} x^4 \ln\left(\dfrac{2-x}{2+x}\right) e^{2x^2-1}\,dx = 0$

Consider now a *periodic* function $f(x)$, with period T:

$$f(x+T) = f(x) \tag{1}$$

Proposition: For any $A \in R$,

$$\boxed{\int_0^T f(x)\,dx = \int_A^{A+T} f(x)\,dx} \tag{2}$$

That is, *the integral of a periodic function has the same value over* <u>*any*</u> *interval equal to a period.*

***Proof*:** We write

$$\int_A^{A+T} f(x)\,dx = \int_A^0 f(x)\,dx + \int_0^T f(x)\,dx + \int_T^{A+T} f(x)\,dx \quad \Rightarrow$$

$$\int_A^{A+T} f(x)\,dx = \int_0^T f(x)\,dx + \int_T^{A+T} f(x)\,dx - \int_0^A f(x)\,dx \qquad (3)$$

But, because of (1),

$$\int_0^A f(x)\,dx = \int_0^A f(x+T)\,dx = \int_0^A f(x+T)\,d(x+T)\,.$$

We make the transformation $u = x+T$, $\int_0^A dx \to \int_T^{A+T} du$. Then,

$$\int_0^A f(x)\,dx = \int_T^{A+T} f(u)\,du = \int_T^{A+T} f(x)\,dx \qquad (4)$$

From (3) and (4) there follows (2).

Examples:

1. $\int_0^{2\pi} \cos x\,dx = [\sin x]_0^{2\pi} = 0$, $\int_{-\pi}^{\pi} \cos x\,dx = [\sin x]_{-\pi}^{\pi} = 0$.

2. $\int_0^{2\pi} \sin x\,dx = -[\cos x]_0^{2\pi} = 0$, $\int_{-\pi}^{\pi} \sin x\,dx = -[\cos x]_{-\pi}^{\pi} = 0$.

3. $\int_0^{\pi} \cos 2x\,dx = \frac{1}{2}[\sin 2x]_0^{\pi} = 0$, $\int_{-\pi/2}^{\pi/2} \cos 2x\,dx = \frac{1}{2}[\sin 2x]_{-\pi/2}^{\pi/2} = 0$.

4. $\int_0^{\pi} \sin 2x\,dx = -\frac{1}{2}[\cos 2x]_0^{\pi} = 0$, $\int_{-\pi/2}^{\pi/2} \sin 2x\,dx = -\frac{1}{2}[\cos 2x]_{-\pi/2}^{\pi/2} = 0$.

5.4 Integrals with Variable Limits

As we know, the *indefinite* integral of a function represents the *infinite set* of antiderivatives of this function, while the *definite* integral with *constant* limits of integration (upper and lower) is just a real number. But, what if we allow one of the limits of a definite integral – say, the upper limit – to be *variable*? In this case the integral will no longer be a constant, since its value will depend on the value of the upper limit. In other words, the integral will be a function of its upper limit.

Making a slight change to our previous notation, we put *t* in place of *x* and we denote by *x* the variable upper limit of the integral. Given a function $f(t)$ we then define the following function of *x*:

$$I(x) = \int_a^x f(t)\,dt \qquad (1)$$

Theorem: The function $I(x)$ is an antiderivative of the function $f(x)$:

$$\boxed{I'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

Proof: Let $F(x)$ be an arbitrary antiderivative of $f(x)$: $F'(x)= f(x)$. Obviously, $F(t)$ will then be an antiderivative of $f(t)$: $F'(t)= f(t)$. Therefore,

$$I(x) = \int_a^x f(t)\,dt = [F(t)]_a^x = F(x) - F(a) \;\Rightarrow\; I'(x) = F'(x) - 0 = f(x) \ .$$

Comment: The function $I(x)$ does not depend on the choice of the antiderivative $F(x)$. Indeed, if $G(x)=F(x)+C$ is any other antiderivative of $f(x)$, then

$$I(x) = [F(t)]_a^x = [F(t)+C]_a^x = [G(t)]_a^x = G(x) - G(a) \ .$$

Example: Let $f(x) = x^2 \Rightarrow f(t) = t^2$. We define

$$I(x) = \int_a^x f(t)\,dt = \int_a^x t^2 dt = [t^3/3]_a^x = (x^3/3) - (a^3/3) \ .$$

Then, $I'(x) = x^2 = f(x)$.

Now, let us go one step further by assuming that, in addition to the upper limit of an integral, the *lower* limit is variable as well. In this case the integral $I(x)$ in relation (1) does not represent a specific antiderivative of $f(x)$ but, rather, a whole *infinity* of antiderivatives, each one corresponding to a certain value of the lower limit. In other words, $I(x)$ in (1) is an *indefinite integral*! We write, by omitting the lower-limit symbol (since this limit is unspecified anyway):

$$I(x) = \int^x f(t)\,dt \equiv \int f(x)\,dx = F(x) + C \ , \quad \text{where } F'(x)= f(x) \ .$$

5.5 Improper Integrals: Infinite Limits

A definite integral is *proper* if (*a*) the interval of integration $[a, b]$ is *closed* and *finite* (neither of the a and b is infinite) and (*b*) the function to be integrated (the *integrand*) takes on *finite values* everywhere within $[a, b]$. If even one of these conditions is not satisfied, the integral is called *improper*.

We begin our study of improper integrals by examining the case of infinite intervals of integration. Such integrals are defined as follows:

$$\int_a^{+\infty} f(x)\,dx \equiv \lim_{b\to+\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx \equiv \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx \equiv \lim_{\substack{b\to+\infty \\ a\to-\infty}} \int_a^b f(x)\,dx$$

If the limit exists and is finite, we say that the corresponding improper integral *converges* (is *convergent*). If the limit does not exist, or if it is infinite, the corresponding integral *diverges* (is *divergent*).

Let $F(x)$ be an antiderivative of $f(x)$. Then, for finite a and b,

$$\int_a^b f(x)\,dx = F(b) - F(a) \ .$$

In the cases of infinite limits this is extended as follows:

1. $$I = \int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} F(b) - F(a) \ ;$$

the integral I converges if the limit of $F(b)$ exists and is finite.

2. $$I = \int_{-\infty}^{b} f(x)\,dx = F(b) - \lim_{a\to-\infty} F(a) \ ;$$

the integral I converges if the limit of $F(\alpha)$ exists and is finite.

3. $$I = \int_{-\infty}^{+\infty} f(x)\,dx = \lim_{b\to+\infty} F(b) - \lim_{a\to-\infty} F(a) \ ;$$

the integral I converges if the limits of both $F(a)$ and $F(b)$ exist and are finite (if either limit does not exist or is infinite, I diverges). Alternatively, we can write I as a sum of improper integrals:

$$I = \int_{-\infty}^{0} f(x)\,dx + \int_0^{+\infty} f(x)\,dx = \int_0^{+\infty} f(-x)\,dx + \int_0^{+\infty} f(x)\,dx \qquad (1)$$

(Notice that

$$\int_{-\infty}^{0} f(x)\,dx = -\int_{-\infty}^{0} f\big(-(-x)\big)\,d(-x) = -\int_{+\infty}^{0} f(-u)\,du = \int_0^{+\infty} f(-x)\,dx \ ,$$

where in the last step we just renamed the integration variable from u to x.) The integral I converges if both integrals on the right-hand side of (1) converge.

Careful! It is generally *wrong* to define I as

$$I=\int_{-\infty}^{+\infty} f(x)\,dx=\lim_{l\to+\infty}\int_{-l}^{+l} f(x)\,dx=\lim_{l\to+\infty}[F(l)-F(-l)] \quad (wrong\,!!!)$$

The reason is the following: In order for I to converge, the limits of *both* $F(l)$ and $F(-l)$ must exist for $l\to+\infty$. Now, imagine that $F(l)$ is an even function that becomes infinite at $\pm\infty$. Then obviously I *diverges*. On the other hand, since $F(l)$ is even we have that $F(l)-F(-l)=0$. Thus, if we adopted the aforementioned *erroneous* definition of I we would come to the *wrong* conclusion that $I=0$, i.e., that I converges!

Examples:

1. $I=\int_{-\infty}^{+\infty}\frac{x\,dx}{1+x^2}$.

We have: $\int_a^b \frac{x\,dx}{1+x^2}=\frac{1}{2}\,[\ln(1+x^2)]_a^b=\frac{1}{2}\{\ln(1+b^2)-\ln(1+a^2)\}$. Then,

$I=\frac{1}{2}\,\{\lim_{b\to+\infty}[\ln(1+b^2)]-\lim_{a\to-\infty}[\ln(1+a^2)]\}$. Both limits are infinite, hence I diverges.

2. $I=\int_0^{+\infty}\cos x\,dx$.

We have: $I=\lim_{b\to+\infty}\int_0^b \cos x\,dx=\lim_{b\to+\infty}(\sin b)$.

We observe that, as b tends to infinity, $\sin b$ "oscillates" endlessly between -1 and $+1$, never attaining a fixed value! Thus the limit of $\sin b$ does not exist and I diverges.

3. $I=\int_{-\infty}^{+\infty}\frac{dx}{1+x^2}$.

We have:

$$I=\lim_{\substack{b\to+\infty\\ a\to-\infty}}\int_a^b \frac{dx}{1+x^2}=\lim_{\substack{b\to+\infty\\ a\to-\infty}}[\arctan x]_a^b=\lim_{b\to+\infty}(\arctan b)-\lim_{a\to-\infty}(\arctan a)=\frac{\pi}{2}-(-\frac{\pi}{2})\Rightarrow$$

$$\boxed{\int_{-\infty}^{+\infty}\frac{dx}{1+x^2}=\pi}$$

Exercise 5.4 Let $I=\int_0^{+\infty} e^{ax}dx$. Show that I diverges for $a\geq 0$ and converges for $a<0$. In the latter case show that

$$\boxed{\int_0^{+\infty} e^{-kx}dx=\frac{1}{k}\quad (k>0)}$$

Exercise 5.5 Show that the integral $I=\int_{-\infty}^{+\infty} e^{ax} dx$ diverges for all values of a. (*Hint:* Write I as a sum of two integrals from 0 to $+\infty$ and notice that one of these integrals must diverge.)

Exercise 5.6 Let $I=\int_1^{+\infty} \frac{dx}{x^k}$. Show that I converges for $k>1$ and diverges for $k\leq 1$.

Theorem (comparison test): Consider the integrals

$I_1=\int_a^{+\infty} f(x)\,dx$, $I_2=\int_a^{+\infty} g(x)\,dx$ $(a\in R)$, where $0\leq f(x)\leq g(x), \forall\, x\in[a,+\infty)$.

The following can be proven [1]:

- If I_2 *converges* then I_1 also *converges*.
- If I_1 *diverges* then I_2 also *diverges*.

Examples:

1. Let $I=\int_{-\infty}^{+\infty} e^{-x^2} dx$.

Since the integrand is an even function, we have: $I=2\int_0^{+\infty} e^{-x^2} dx$.

Now, $\int_0^{+\infty} e^{-x^2} dx=\int_0^1 e^{-x^2} dx+\int_1^{+\infty} e^{-x^2} dx$, where the first integral obviously converges. We need to check the second integral for convergence.

We consider the integrals $I_1=\int_1^{+\infty} e^{-x^2} dx$, $I_2=\int_1^{+\infty} e^{-x} dx$.

In the interval $[1, +\infty)$ we have that $e^{-x^2}\leq e^{-x}$ (show this!). Moreover, I_2 converges and equals $I_2=1/e$ (show!). Therefore I_1 converges and hence so does the given integral I. As can be proven,

$$\int_{-\infty}^{+\infty} e^{-x^2} dx=\sqrt{\pi} \ .$$

2. Let $I=\int_1^{+\infty} \frac{\sqrt{x}}{1+x}\, dx$.

In the interval of integration (i.e. for $x\geq 1$) we have that $\frac{\sqrt{x}}{1+x}>\frac{\sqrt{x}}{2x}=\frac{1}{2\sqrt{x}}$.

On the other hand, the integral $\int_1^{+\infty} \frac{dx}{\sqrt{x}}=\int_1^{+\infty} \frac{dx}{x^{1/2}}$ diverges (see Exercise 5.6).

We conclude that the given integral I diverges.

Theorem (absolute convergence): Consider the integrals

$$I_1 = \int_a^{+\infty} f(x)\,dx \;, \quad I_2 = \int_a^{+\infty} |\,f(x)\,|\,dx \quad (a \in R) \;.$$

As can be proven [1], if I_2 *converges* then I_1 also *converges*. We say that I_1 is *absolutely convergent*. (If I_2 diverges, I_1 may or may not converge. Obviously, if I_1 diverges then I_2 also diverges.)

Example: We show that $I = \int_0^{+\infty} \frac{\cos x}{1+x^2}\,dx$ converges.

It suffices to show that I is absolutely convergent, i.e. that $I_1 = \int_0^{+\infty} \frac{|\cos x|}{1+x^2}\,dx$ converges. Indeed, we have that $\frac{|\cos x|}{1+x^2} \le \frac{1}{1+x^2}$, as well as that $\int_0^{+\infty} \frac{dx}{1+x^2} = [\arctan x]_0^{+\infty} = \pi/2 - 0 = \pi/2$ (converges).

By the comparison test, I_1 converges; hence so does the given I.

Exercise 5.7 Show similarly that the integral $I = \int_0^{+\infty} \frac{\sin x}{1+x^2}\,dx$ converges.

Comment: The integral $\int_0^{+\infty} |\cos x|\,dx$ assumes an infinite value, hence diverges. As we saw earlier, the integral $\int_0^{+\infty} \cos x\,dx$ also diverges, albeit in a different sense (explain).

5.6 Improper Integrals: Unbounded Integrand

A different case of improper integral is that where the interval of integration [a, b] is finite but the integrand itself becomes infinite at either limit a or b (or perhaps at both).

Definition:

1. Let $f(x)$ be continuous in the interval [a, b) but become infinite for $x \to b$. Then,

$$\int_a^b f(x)\,dx \equiv \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx \quad (\varepsilon > 0)$$

provided that the limit exists.

2. Let $f(x)$ be continuous in the interval $(a, b]$ but become infinite for $x \to a$. Then,

$$\int_a^b f(x)\,dx \equiv \lim_{\delta \to 0} \int_{a+\delta}^b f(x)\,dx \quad (\delta > 0)$$

provided that the limit exists.

3. Let $f(x)$ be continuous in the interval (a, b) but become infinite for $x \to a$ and for $x \to b$. Then,

$$\int_a^b f(x)\,dx \equiv \lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \int_{a+\delta}^{b-\varepsilon} f(x)\,dx \quad (\varepsilon > 0,\ \delta > 0)$$

provided that both limits exist.

4. Let $f(x)$ be continuous in the intervals $[a, c)$ and $(c, b]$ but become infinite for $x \to c$ $(a < c < b)$. We write

$$I = \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \equiv I_1 + I_2 \ .$$

The integral I will converge if both I_1 and I_2 converge.

In any case, if the improper integral is convergent we write:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

Examples:

1. $\displaystyle\int_0^a \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_0^a = 2\sqrt{a} \quad (a > 0)$,

despite the fact that the integrand becomes infinite at the lower limit.

2. $\displaystyle\int_0^1 \frac{dx}{1-x} = [-\ln(1-x)]_0^1 \;\Rightarrow$ becomes infinite for $x \to 1$. Thus the integral diverges.

3. $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = [\arcsin x]_{-1}^{1} = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$,

despite the fact that the integrand becomes infinite at both limits.

4. $\int_{-1}^{2} \frac{dx}{\sqrt[3]{x^2}} = \int_{-1}^{0} \frac{dx}{\sqrt[3]{x^2}} + \int_{0}^{2} \frac{dx}{\sqrt[3]{x^2}} = [3\sqrt[3]{x}]_{-1}^{0} + [3\sqrt[3]{x}]_{0}^{2} = 3 + 3\sqrt[3]{2}$.

5. $\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2} \quad \Rightarrow$ both integrals diverge .

Exercise 5.8 Show that the integrals

$$\int_a^b \frac{dx}{(x-a)^k} \quad \text{and} \quad \int_a^b \frac{dx}{(b-x)^k}$$

converge for $k<1$ and diverge for $k \geq 1$.

5.7 The Definite Integral as a Plane Area

By using the definite integral we may calculate areas of domains of the *xy*-plane, bounded by graphs of functions.

Theorem 1: Let $f(x)$ be continuous in the interval $[a\,,b]$, and let $f(x) \geq 0 \ \forall x \in [a\,,b]$ (see Fig. 5.1). Then the area of the plane domain R bounded by the graph of $f(x)$, the x-axis and the lines $x=a$ and $x=b$, is given by the integral

$$A = \int_a^b f(x)\,dx \ .$$

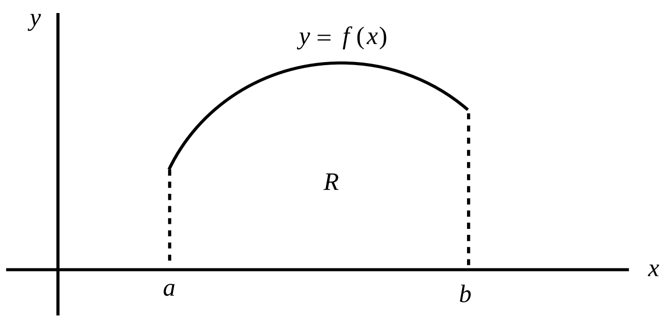


Fig. 5.1. A plane domain R bounded by the graph of $y=f(x)$.

Theorem 2: Let $f(x)$ and $g(x)$ be continuous in the interval $[a\,,b]$, and let $f(x) \geq g(x)$ $\forall x \in [a\,,b]$ (see Fig. 5.2). Then the area of the plane domain R bounded by the graphs of $f(x)$ and $g(x)$ and the lines $x=a$ and $x=b$, is given by the integral

$$A = \int_a^b \big(f(x) - g(x) \big)\,dx \ .$$

(Notice that, for $g(x) \equiv 0$ the graph of $g(x)$ is a part of the x-axis and thus Theorem 2 reduces to Theorem 1.)

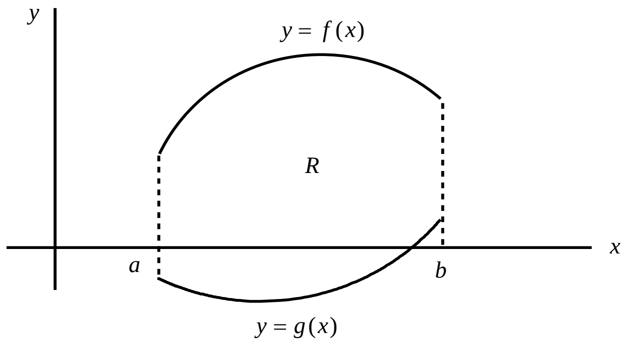


Fig. 5.2. A plane domain R bounded by the graphs of $y=f(x)$ and $y=g(x)$.

Corollary: The variable area

$$A(x) = \int_a^x f(t)\,dt$$

is an *antiderivative* of $f(x)$: $A'(x) = f(x)$ (explain).

Note 1: If $g(x)$ is continuous in $[a, b]$ and if $g(x) \le 0 \;\forall x \in [a, b]$, then the area of the plane domain R bounded by the graph of $g(x)$, the x-axis and the lines $x=a$ and $x=b$ is equal to

$$A = -\int_a^b g(x)\,dx = \int_a^b |g(x)|\,dx \;.$$

Note 2: The area of the plane domain bounded by the graphs of $f(x)$ and $g(x)$ for $a \le x \le b$ is equal to

$$A = \int_a^b |f(x) - g(x)|\,dx \;,$$

regardless of the sign of the difference $f(x) - g(x)$ for the various values of x!

Example: Find the area of the domain bounded by the graph of $f(x) = x^3$ and the x-axis, for $-1 \le x \le 1$.

Solution: We notice that $f(x) \le 0$ for $x \in [-1, 0]$ and $f(x) \ge 0$ for $x \in [0, 1]$. Thus,

$$A = \int_{-1}^{1} |x^3|\,dx = \int_{-1}^{0} |x^3|\,dx + \int_{0}^{1} |x^3|\,dx = -\int_{-1}^{0} x^3 dx + \int_{0}^{1} x^3 dx$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \;.$$

Exercise 5.9 Imagine that the graph in Fig. 5.1 is displaced to the right by $\Delta x = c$. Show that the area of the new plane domain R' between the displaced curve and the x-axis will be the same as that of the original domain R. [*Hint:* Notice that the new curve extends from $a+c$ to $b+c$ and is described by the function $y=h(x)=f(x-c)$.]

Reference

1. A. F. Bermant, I. G. Aramanovich, *Mathematical Analysis* (Mir Publishers, 1975).

CHAPTER 6

SERIES

6.1 Series of Constants

Let $a_n = a_1, a_2, a_3, ...,$ be an infinite sequence of real numbers. The infinite sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

is called a *numerical series*. The number a_n is the *general term* of the series. To construct the series we need to be given a rule f according to which $a_n = f(n)$ (n=1,2,3,...).

Examples:

1. For $a_n = f(n) = \dfrac{1}{2^n} \Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \dfrac{1}{2^n} = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots$.
2. For $a_n = f(n) = \dfrac{1}{n!} \Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \dfrac{1}{n!} = 1 + \dfrac{1}{2} + \dfrac{1}{6} + \cdots$.

The sum of the first n terms, $S_n = a_1 + a_2 + \ldots + a_n$, is called the nth *partial sum* of the series. For n=1,2,3,..., the partial sums themselves form an infinite sequence:

$$S_1 = a_1 \;,\quad S_2 = a_1 + a_2 \;,\; \ldots \;,\quad S_n = a_1 + a_2 + \ldots + a_n \;,\; \ldots$$

If this sequence converges to a finite limit s as $n \to \infty$, i.e., if $\lim_{n\to\infty} S_n = s \in R$, we say that the series *converges* (is *convergent*) and the number s is the *sum* of the series. We write

$$s = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$$

If the limit of the sequence S_n is infinite or does not exist at all, the series *diverges* (is *divergent*).

Example: The *geometrical series* is written

$$\sum_{n=1}^{\infty} \alpha q^{n-1} = \alpha + \alpha q + \alpha q^2 + \cdots \quad (\alpha \neq 0) \;.$$

That is, $a_1 = \alpha$, $a_2 = \alpha q$, $a_3 = \alpha q^2$, ... , $a_n = \alpha q^{n-1}$, ... The nth partial sum is

$$S_n = \alpha + \alpha q + \alpha q^2 + \cdots + \alpha q^{n-1} = \alpha \frac{q^n - 1}{q - 1} \quad \text{if} \quad q \neq 1$$
$$= \quad n\alpha \quad \text{if} \quad q = 1$$

We have the following cases:

1. If $|q| > 1$ then $q^n \to \pm\infty$, S_n becomes infinite and the series *diverges*.

2. If $q = 1$ then $S_n = n\alpha \to \infty$ and the series *diverges*.

3. If $q = -1$, the value of S_n alternates between α and 0 as $n \to \infty$, so that S_n does not tend to any definite limit. Hence the series *diverges*.

4. If $|q| < 1$ (i.e., $-1 < q < 1$) then $q^n \to 0$ and $S_n \to \alpha/(1-q)$, which is a finite limit. Thus the series *converges*, its sum being equal to

$$\boxed{\sum_{n=1}^{\infty} \alpha q^{n-1} = \frac{\alpha}{1-q} \qquad (|q| < 1)}$$

In conclusion,

the geometrical series converges for $|q| < 1$ *and diverges for* $|q| \geq 1$.

Theorem (necessary condition for convergence):

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\boxed{\lim_{n\to\infty} a_n = 0}$.

Careful: This condition is *necessary* but *not sufficient* for convergence! That is, the fact that $a_n \to 0$ does *not* imply that the series must converge!

Corollary: If $\lim_{n\to\infty} a_n \neq 0$ then the series *diverges*.

Examples:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{100n+1} = \frac{1}{101} + \frac{2}{201} + \frac{3}{301} + \cdots$.

We have: $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{100n+1} = \lim_{n\to\infty} \frac{1}{100 + \frac{1}{n}} = \frac{1}{100} \neq 0 \Rightarrow$ the series diverges.

2. As can be proven, the *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ *diverges* (its sum is infinite) despite the fact that $a_n = 1/n \to 0$!

Note: More generally, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} = 1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + \cdots$$

converges for $\alpha > 1$ and diverges for $\alpha \le 1$.

6.2 Positive Series

In this section we consider series $\sum_{n=1}^{\infty} a_n$ with $a_n > 0$, $\forall\, n$ (*positive series*).

Theorem (comparison test):

Consider the series $A \equiv \sum_{n=1}^{\infty} a_n$ and $B \equiv \sum_{n=1}^{\infty} b_n$ where $0 < a_n \le b_n$, $\forall\, n$.
The following can be proven [1]:

- If B *converges* then A also *converges*.
- If A *diverges* then B also *diverges*.

Examples:

1. Let $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$.

We notice that, for $n > 1$, $\sqrt{n} < n \;\Rightarrow\; \frac{1}{\sqrt{n}} > \frac{1}{n}$.

Moreover, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Thus the given series diverges.

2. Let $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots$.

We notice that $\frac{1}{n \cdot 2^n} < \frac{1}{2^n}$. Moreover, the geometrical series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{2} \right)^{n-1}$ converges (why?). Thus the given series converges.

Theorem (D'Alembert's test):

Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$, $\forall\, n$. We call $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

The following can be proven [1]:

- If $\rho < 1$ the series *converges*.
- If $\rho > 1$ the series *diverges*.
- If $\rho = 1$ the test *fails*.

Example: Let $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots$.

We have: $a_n = \frac{n}{2^n}$, $a_{n+1} = \frac{n+1}{2^{n+1}}$, $\frac{a_{n+1}}{a_n} = \frac{1}{2}\frac{n+1}{n} = \frac{1}{2}\left(1+\frac{1}{n}\right) \underset{n\to\infty}{\to} \frac{1}{2} < 1$.

Thus the given series converges.

6.3 Absolutely Convergent Series

A series $A \equiv \sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if the corresponding positive series $A' \equiv \sum_{n=1}^{\infty} |\, a_n \,|$ converges.

Theorem: If a series is absolutely convergent, then it is convergent [1]. (That is, if the series A' of absolute values converges, then the series A itself also converges.)

The converse of this theorem is *not* true: a convergent series is *not necessarily* absolutely convergent also. For example, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

converges, while the (harmonic) series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Example: Let $A \equiv \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$.

The corresponding series of absolute values,

$$A' \equiv \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^{n-1}$$

is a convergent geometrical series. Thus, being absolutely convergent, the given series A is convergent.

Exercise 6.1 By using D'Alembert's test, verify that the above series A' of absolute values converges.

Exercise 6.2 Find the sums of the series A and A' of the above example. (*Hint:* Notice that both series are geometrical.)

6.4 Functional Series

Series whose terms are *functions* rather than constant numbers are called *functional series*. The general form of a functional series is

$$\sum_{n=1}^{\infty} a_n(x) = a_1(x) + a_2(x) + \cdots \quad .$$

This series may converge for some values of x and diverge for others. A point $x=x_0$ at which the *numerical* series $a_1(x_0) + a_2(x_0) + \ldots$ converges is called *point of convergence* of the series. The set of all points of convergence is called *domain of convergence* of the series. The *sum* of a functional series is a function of x, defined in the domain of convergence of the series:

$$s(x) = \sum_{n=1}^{\infty} a_n(x) = a_1(x) + a_2(x) + \cdots \quad .$$

Example: Consider the geometrical series

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots \quad .$$

This series converges in the interval $(-1, 1)$, given that for every $x=x_0$ in that interval the corresponding numerical series $1 + x_0 + x_0^2 + \ldots$ converges. The sum of the series in the domain of convergence is

$$\boxed{\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad , \quad x \in (-1, 1)}$$

For $x \notin (-1, 1)$ the series *diverges* and its sum cannot be defined.

Example: We will find the domain of convergence of the series

$$A \equiv \sum_{n=1}^{\infty} \frac{\sin nx}{n^2} = \sin x + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \cdots \quad .$$

We consider the series of absolute values, $A' \equiv \sum_{n=1}^{\infty} \frac{|\sin nx|}{n^2}$. We notice that $\frac{|\sin nx|}{n^2} \leq \frac{1}{n^2}$, $\forall x \in R$. Taking into account that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we conclude that the series A' converges $\forall x \in R$. This means that the original series A is absolutely convergent and thus convergent $\forall x \in R$. Therefore, the domain of convergence of A is R.

Example: Show that the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{1}$$

converges for all $x \in R$. By using this result, show that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \ , \quad \forall x \in R \tag{2}$$

Solution: We consider the series of absolute values, $\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$. By putting $a_n = |x|^n / n!$ we notice that $a_{n+1}/a_n = |x| / (n+1) \to 0 < 1$, $\forall x \in R$. Thus, by D'Alembert's criterion this series converges $\forall x \in R$. This means that the given series (1) is absolutely convergent, thus convergent $\forall x \in R$. Relation (2) then simply expresses the condition for convergence of the series (1), which condition is here satisfied.

Exercise 6.3 Show that the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

converges for $|x| < 1$ $(-1 < x < 1)$. [*Hint:* Consider the series of absolute values and use D'Alembert's test to show that this series converges in the interval $(-1, 1)$.]

6.5 Expansion of Functions into Power Series

A *power series* is a functional series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots \tag{1}$$

The constants a_n are the *coefficients* of the power series. In particular, for $x_0 = 0$,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{2}$$

We note that *every* power series can be written in the form (2) by making the substitution $x - x_0 = x'$.

We consider a power series of the form (2). We assume that a number *r* exists such that the series converges for $|x| < r$ and diverges for $|x| > r$ (the series may converge or diverge for $x = \pm r$). The number *r* is called the *radius of convergence* of the power series, while the interval $(-r, r)$ is called *interval of convergence* of this series. In particular, if $r = 0$ the series diverges for every $x \neq 0$, while if $r = \infty$ the series converges for every $x \in R$. For a series of the more general form (1) the interval of convergence is written $(x_0 - r,\ x_0 + r)$.

Example: For the geometrical series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

(notice that $\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1}$) the interval of convergence is $(-1, 1)$ and the radius of convergence is $r = 1$.

Problem: Given a function $f(x)$, is it possible to find a convergent power series whose sum equals $f(x)$? Let us see an example: We recall that

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad , \quad \forall x \in (-1, 1)} \tag{3}$$

We observe that, in the interval $(-1, 1)$ (i.e., for $|x| < 1$) and *only* in this interval the function $(1-x)^{-1}$ equals the sum of the geometrical series, in the sense that, for every x in that interval the function and the series assume common values. For $|x| \geq 1$, however, the geometrical series *diverges* while the function $(1-x)^{-1}$ *continues to be defined* (except at the single point $x = 1$)! In any case, the series and the function do *not* assume common values for $|x| \geq 1$.

Generally speaking, if we wish to *expand a function* $f(x)$ *into a power series* of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots \tag{4}$$

or, for $x_0 = 0$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{5}$$

we must be careful to determine the interval D where this expansion makes sense. In that interval the function must be *defined* and the series must *converge* (that is, D must be a subset of both the domain of definition of the function *and* the interval of convergence of the series). The function $f(x)$ itself, however, may still be defined for $x \notin D$, at points where the series *diverges*!

Taylor's theorem: Assume that the differentiable function $f(x)$ may be expanded into a power series of the form (4) in a neighborhood $D = (x_0 - l, x_0 + l)$ of x_0. Then the coefficients a_n of the series are given by the formula

$$\boxed{a_n = \frac{1}{n!} f^{(n)}(x_0)}$$

where $f^{(n)}$ denotes the nth-order derivative of $f(x)$. The series (4) is thus written

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \cdots}$$

and is called the *Taylor series expansion* of $f(x)$ about the point $x = x_0$.

In the (more common) case where $x_0 = 0$, so that $D = (-l, l)$, the power series expansion (5) of $f(x)$ about $x = 0$ is called *Maclaurin's series* and is written

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n = f(0) + f'(0)\, x + \frac{1}{2!} f''(0)\, x^2 + \cdots}$$

A useful alternative form of Taylor's series is found as follows: In the original form of the series, x_0 is constant while x is variable. The difference $h = x - x_0$ is a variable quantity and can be taken as a new variable in place of x. Putting $x = x_0 + h$, we write the Taylor series as follows:

$$\boxed{f(x_0 + h) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)\, h^n = f(x_0) + f'(x_0)\, h + \frac{1}{2!} f''(x_0)\, h^2 + \cdots}$$

For $x_0 = 0$ the above series becomes

$$f(h) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, h^n = f(0) + f'(0)\, h + \frac{1}{2!} f''(0)\, h^2 + \cdots$$

which is the Maclaurin's series (with h in place of x).

Maclaurin series expansions of some functions [1]

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad , \quad D = R$$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \quad , \quad D = R$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad , \quad D = R$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad , \quad D = R$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad , \quad D = (-1\,,1)$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots \quad , \quad D = (-1\,,1)$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad , \quad D = (-1, 1)$$

[1] We denote by *D* the interval within which the expansion is valid.

Exercise 6.4 Prove the relation

$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots}$$

Exercise 6.5 Expand the functions $\sin(-x)$ and $\cos(-x)$ into power series. Show that your results are in agreement with the property of $\sin x$ being odd and $\cos x$ being even.

Exercise 6.6 By using the expansion formula for $(1+x)^{-1}$, prove the expansion formula for $\ln(1+x)$. *Hint:* Notice that

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} \quad .$$

Exercise 6.7 Consider the polynomial

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots + b_n x^n \quad .$$

Show that the Maclaurin expansion of $f(x)$ is the function itself.

Exercise 6.8 For $|x| << 1$ we can make the approximation $x^n \approx 0$ for $n > 1$ (that is, the powers x^2 , x^3 , x^4 , … , are considered negligible for very small values of $|x|$). Show that a function $f(x)$ that can be Maclaurin expanded in an interval $(-a, a)$ may be approximated by

$$f(x) \approx f(0) + f'(0)\,x \quad \text{for} \quad |x| << 1 \quad .$$

As an application, show that for $|x| << 1$,

$$e^x \approx 1 + x \ , \qquad \sin x \approx x \ , \qquad \cos x \approx 1 \quad .$$

Reference

1. A. F. Bermant, I. G. Aramanovich, *Mathematical Analysis* (Mir Publishers, 1975).

CHAPTER 7

AN ELEMENTARY INTRODUCTION TO DIFFERENTIAL EQUATIONS

7.1 Two Basic Theorems

Before we talk about differential equations, it would be useful to state two basic theorems that play a key part in the development of the subject.

Theorem 1: Suppose the following differential relation is true:

$$f(x)\,dx = g(y)\,dy \quad \text{where} \quad y=\varphi(x)\ .$$

Then,

$$\int f(x)\,dx = \int g(y)\,dy\ .$$

(*Careful:* This is equality between *infinite sets*!)

Proof: By the definition of the differential, $dy = d\varphi(x) = \varphi'(x)\,dx$. Thus,

$$f(x)\,dx = g(\varphi(x))\,\varphi'(x)\,dx \;\Rightarrow\; \text{(by eliminating } dx\text{)}$$

$$f(x) = g(\varphi(x))\,\varphi'(x) \;\Rightarrow\; \text{(by integrating identical functions)}$$

$$\int f(x)\,dx = \int g(\varphi(x))\,\varphi'(x)\,dx\ .$$

But, as we saw in Sec. 4.2, the symbol "d" inside the integral has similar properties with the differential of a function. Thus we can set $\varphi'(x)\,dx = d\,\varphi(x)$ inside the integral, so that

$$\int f(x)\,dx = \int g(\varphi(x))\,d\,\varphi(x) = \int g(y)\,dy\,.$$

Theorem 2: Suppose the following differential relation is true:

$$f(x)\,dx = g(y)\,dy \quad \text{where} \quad y=\varphi(x)\ .$$

Moreover, assume that

$$\varphi(x_0) = y_0 \quad (\text{i.e.,}\;\; y=y_0 \;\text{ for }\; x=x_0)\,.$$

Then,

$$\int_{x_0}^{x} f(t)\,dt = \int_{y_0}^{y} g(u)\,du\ .$$

Proof: As we saw earlier, $\int f(x)\,dx = \int g(\varphi(x))\,\varphi'(x)\,dx$.

We rename the variable x as t and we integrate from x_0 to x:

$$\int_{x_0}^{x} f(t)\,dt = \int_{x_0}^{x} g(\varphi(t))\,\varphi'(t)\,dt = \int_{x_0}^{x} g(\varphi(t))\,d\varphi(t) \ .$$

We now make the substitution $u = \varphi(t)$ and we transform the right integral for t into an integral for u. To find the limits of the new integral, we think as follows:

$$\text{for } t = x_0 \ \Rightarrow \ u = \varphi(x_0) = y_0 \ ;$$
$$\text{for } t = x \ \Rightarrow \ u = \varphi(x) = y \ .$$

Thus, $\int_{x_0}^{x} dt \to \int_{y_0}^{y} du$, $\int_{x_0}^{x} g(\varphi(t))\,d\varphi(t) = \int_{y_0}^{y} g(u)\,du$, and therefore,

$$\int_{x_0}^{x} f(t)\,dt = \int_{y_0}^{y} g(u)\,du \ .$$

Note: To simplify our notation we often write

$$\int_{x_0}^{x} f(x)\,dx = \int_{y_0}^{y} g(y)\,dy \ .$$

Note, however, that although the symbols are the same, their roles are different. Indeed, each of the two integrals is *a function of its upper limit*, regardless of the name given to the variable of integration!

7.2 First-Order Differential Equations

We begin with a quick look at the various types of equations of mathematics.

1. An *algebraic equation* is a relation of the form

$$F(x) = 0 \tag{1}$$

where $F(x)$ is some algebraic expression. The *solution* of (1) is the set of values of x (*roots*) that satisfy this equation. The roots of an algebraic equation may be real, complex, or mixed real and complex.

2. A *function* is defined by an equation of the form

$$F(x, y) = 0 \tag{2}$$

Often this relation can be solved for one variable in terms of the other: $y = f(x)$, where to every value of x corresponds a unique value of y (but not necessarily vice versa).

3. A *first-order differential equation* is an equation of the form

$$F(x, y, y') = 0 \quad \text{where} \quad y = y(x) \quad \text{and} \quad y' = dy/dx \tag{3}$$

Often this relation can be solved for the derivative: $y' = f(x, y)$.

The *general solution* of (3) is an *infinite set of functions* $y=y(x)$ that satisfy this equation. The general solution contains an *arbitrary constant parameter C*, thus has the form

$$y = \varphi(x, C) \tag{4}$$

For a specific value $C=C_0$ of the constant we have a *particular solution* of (3):

$$y = \varphi(x, C_0) \tag{5}$$

To determine such a particular solution, in addition to the differential equation we must be given an *initial condition*, in the form

$$y = y_0 \quad \text{when} \quad x = x_0 \quad \Leftrightarrow \quad y(x_0) = y_0 \tag{6}$$

By substituting the information (6) into the general solution (4) we get an algebraic equation of the form $y_0 = \varphi(x_0, C)$. Solving this for C we can determine the constant C_0 and thus find the particular solution (5).

The process of finding the general or some particular solution of a differential equation is called *integration* of the differential equation.

7.3 Some Special Cases

Let us see some special cases of differential equations of the form $y' = f(x, y)$.

1. We begin with the most trivial case, which, however, is pedagogically useful for understanding the general philosophy of solving differential equations:

$$y' = f(x) \tag{1}$$

$$\text{with initial condition} \quad y = y_0 \quad \text{when} \quad x = x_0 \ . \tag{2}$$

We can work in two ways:

(*a*) We find the *general* solution of (1), which will contain an arbitrary constant, and then use the initial condition (2) in order to determine the value of this constant and the respective *particular* solution. We thus write, taking into account Theorem 1 of Sec. 7.1:

$$\frac{dy}{dx} = f(x) \;\Rightarrow\; dy = f(x)\,dx \;\Rightarrow\; \int dy = \int f(x)\,dx \ .$$

If $F(x)$ is an arbitrary antiderivative of $f(x)$, then $y + C_1 = F(x) + C_2 \;\Rightarrow$

$$y(x) = F(x) + C \quad \text{(general solution)} \tag{3}$$

We now apply the initial condition (2) to (3):

$$y_0 = F(x_0) + C \;\Rightarrow\; C = y_0 - F(x_0)\,.$$

Substituting for C into (3), we find $y = F(x) + y_0 - F(x_0) \;\Rightarrow$

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt} \quad \text{(particular solution)} \tag{4}$$

(b) We find the particular solution *directly* [without finding the general solution (3) first], taking into account Theorem 2 of Sec. 7.1:

$$\frac{dy}{dx} = f(x) \;\Rightarrow\; dy = f(x)\,dx \;\Rightarrow\; \int_{y_0}^{y} du = \int_{x_0}^{x} f(t)\,dt \;\Rightarrow\; (4)\,,\ \text{as before}\,.$$

This approach, although shorter and perhaps more suitable for practical applications, has the drawback of not giving us any information regarding the general solution of the differential equation.

Exercise 7.1 Verify that the particular solution (4) satisfies the differential equation (1) as well as the initial condition (2). (*Hint:* Notice that y is a function of the upper limit x of the integral.)

Note: We usually simplify the notation in (4) by discarding the auxiliary symbol t and simply writing x in its place:

$$y(x) = y_0 + \int_{x_0}^{x} f(x)\,dx\,.$$

We should not forget, however, that y is *a function of the upper limit* of the integral, regardless of the name given to the variable of integration!

2. Consider the so-called *separable differential equation*:

$$y' = f(x)\,g(y) \tag{5}$$

$$\text{with initial condition} \quad y = y_0 \ \text{when} \ x = x_0\,.$$

Due to the special form of the right-hand side of (5), the variables x and y can be *separated* so that y may appear only on the left-hand side while x appears on the right-hand side. We use the second method, which gives the particular solution directly:

$$\frac{dy}{dx} = f(x)\,g(y) \;\Rightarrow\; \frac{dy}{g(y)} = f(x)\,dx \;\Rightarrow$$

$$\boxed{\int_{y_0}^{y} \frac{dy}{g(y)} = \int_{x_0}^{x} f(x)\,dx} \quad \text{(particular solution)} \tag{6}$$

Exercise 7.2 Verify that the expression (6) satisfies the differential equation (5). [*Hint:* Differentiate both sides with respect to *x*. Notice that in the right integral *x* is an upper limit, while in the left integral *x* "appears" in the upper limit through *y*. Consider thus the left integral as a composite function to be differentiated first for *y* (the one in the upper limit) and then for *x*.]

Exercise 7.3 Find expressions analogous to (6) for the particular solutions of the following separable differential equations:

(1) $y' = g(y)$ (2) $y' = f(x)/g(y)$ (3) $y' = g(y)/f(x)$

7.4 Examples

Let us see some examples of differential equations with given initial conditions.

1. $y' = ay \quad | \quad y = y_0$ when $x = x_0$.

We find the general solution (assuming that $y > 0$, $\forall x$) :

$$\frac{dy}{dx} = ay \;\Rightarrow\; \frac{dy}{y} = a dx \;\Rightarrow\; \int \frac{dy}{y} = a\int dx \;\Rightarrow\; \ln y + C_1 = ax + C_2 \;\Rightarrow$$

$$\ln y = ax + C \;\Rightarrow\; y = e^{ax+C} \;\Rightarrow\; y = C e^{ax} \quad \text{(general solution)}$$

where in the last step we put C in place of e^C . To apply the initial condition, we set x=x_0 and y=y_0 in the general solution and we solve for *C*. The result is $C = y_0\, e^{-a x_0}$. Thus the particular solution is

$$y = y_0\, e^{a(x-x_0)} .$$

We can find the particular solution directly (without using the general solution) as follows:

$$\frac{dy}{y} = a\,dx \;\Rightarrow\; \int_{y_0}^{y} \frac{dy}{y} = a\int_{x_0}^{x} dx \;\Rightarrow\; \ln\left(\frac{y}{y_0}\right) = a(x - x_0) \;\Rightarrow\; y = y_0\, e^{a(x-x_0)} .$$

2. $y' = 3x^2 y \quad | \quad y = 2$ when $x = 0$.

We find the particular solution directly:

$$\frac{dy}{dx} = 3x^2 y \;\Rightarrow\; \frac{dy}{y} = 3x^2 dx \;\Rightarrow\; \int_2^y \frac{dy}{y} = 3\int_0^x x^2 dx \;\Rightarrow\; \ln(y/2) = x^3 \;\Rightarrow\; y = 2e^{x^3} .$$

Exercise: Find the general solution of the differential equation and show that it yields the same particular solution for the given initial condition.

3. $y' = x^3 e^{-y}$ | $y = 0$ when $x = 0$.

$$\frac{dy}{dx} = x^3 e^{-y} \Rightarrow e^y dy = x^3 dx \Rightarrow \int_0^y e^y dy = \int_0^x x^3 dx \Rightarrow e^y - 1 = \frac{x^4}{4} \Rightarrow$$
$$y = \ln\left(\frac{x^4}{4} + 1\right) .$$

Exercise: Verify that this solution satisfies both the differential equation and the initial condition.

4. $y' = -x^3 / y^3$ (general solution only) .

$$\frac{dy}{dx} = -\frac{x^3}{y^3} \Rightarrow y^3 dy = -x^3 dx \Rightarrow \int y^3 dy = -\int x^3 dx \Rightarrow \frac{y^4}{4} = -\frac{x^4}{4} + C_1 \Rightarrow$$
$$y^4 + x^4 + C = 0 .$$

Notice that the solution is an *implicit* function (Sec. 1.4).

Exercise: Verify that the above solution satisfies the given differential equation. [*Hint:* Differentiate the solution with respect to x (cf. Sec. 2.12).]

CHAPTER 8

INTRODUCTION TO DIFFERENTIATION IN HIGHER DIMENSIONS

8.1 Partial Derivatives and Total Differential

So far we have studied functions $f(x)$ of a single independent variable x. In this chapter we consider functions of *several* variables. For simplicity, we restrict ourselves to functions of two independent variables only, called x and y. These functions are of the form $u = f(x, y)$, where u is the dependent variable.

The function $u=f(x,y)$, to be written more simply as $u=u(x,y)$, can be differentiated *separately* for x and for y, this process yielding two *partial derivatives* of u. To find the partial derivative with respect to x, we simply differentiate u for x *while treating y as if it were a constant*. Similarly, to find the partial derivative of u with respect to y we must keep x constant. For the partial derivatives we use the symbols $\partial u/\partial x$ and $\partial u/\partial y$. In a formal sense, we define these derivatives as follows:

$$\frac{\partial u}{\partial x}(x,y) \equiv \lim_{\Delta x\to 0}\frac{u(x+\Delta x,\ y)-u(x,y)}{\Delta x}$$
$$\frac{\partial u}{\partial y}(x,y) \equiv \lim_{\Delta y\to 0}\frac{u(x,\ y+\Delta y)-u(x,y)}{\Delta y}$$

We also introduce the *partial differential operators* $\partial/\partial x$ and $\partial/\partial y$:

$$\frac{\partial}{\partial x}u(x,y) \equiv \frac{\partial u(x,y)}{\partial x}\ ,\quad \frac{\partial}{\partial y}u(x,y) \equiv \frac{\partial u(x,y)}{\partial y}\quad .$$

Examples:

1. Let $u(x,y) = x^3\cos 2y$. Then, $\partial u/\partial x = 3x^2\cos 2y$, $\partial u/\partial y = -2x^3\sin 2y$.

2. Let $u(x,y) = (x^2+y^2)^3$. Then, $\partial u/\partial x = 6x(x^2+y^2)^2$, $\partial u/\partial y = 6y(x^2+y^2)^2$.

Higher-order partial derivatives may also be defined. For example,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\ ,\quad \frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\ ,\quad \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\ ,\quad \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\ .$$

According to a theorem of advanced mathematical analysis, if u and its partial derivatives are continuous functions then the two “mixed” partial derivatives on the right are equal:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad .$$

This means that the differentiation of u for x *and* y always yields a unique result independently of the order in which the partial differentiations for x and y are performed.

Exercise 8.1 Verify the above statement for $u(x,y) = \cos(x^2+y^2 - xy)$.

The concept of the differential of a function of a single variable (Sec. 2.7) can be extended to functions of two (or more) variables. We thus define the *total differential* of $u(x,y)$ by the expression

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy$$

where $dx=\Delta x$ and $dy=\Delta y$ are the changes of x and y. [In general, however, the differential du is *not* the same as the change $\Delta u = u(x+\Delta x\,,\, y+\Delta y) - u(x,y)$, unless the function $u(x, y)$ is linear (i.e., does not contain powers or products of x and y) or unless the changes Δx and Δy are infinitesimal.]

Example: For $u(x,y) = x^2 \ln y$ $(y>0)$ we have: $du = (2x \ln y)\, dx + (x^2/y)\, dy$.

Exercise 8.2 Find the total differential du of $u(x,y) = (x^3 - y^3)\, e^{xy}$.

8.2 Exact Differential Equations

A first-order differential equation $dy/dx = f(x, y)$ can always be put in the form

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \quad (N \neq 0)$$

for suitable functions M and N. This is written more symmetrically as

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1}$$

The differential equation (1) is said to be *exact* if there exists a function $u(x, y)$ such that the left-hand side $Mdx+Ndy$ is the total differential of u:

$$M(x, y)\, dx + N(x, y)\, dy = du(x,y) \tag{2}$$

Then, by (1) and (2), $du=0 \;\Rightarrow$

$$u(x, y) = C \tag{3}$$

where C is some constant. Equation (3) is an algebraic relation connecting x and y and containing an arbitrary constant. Thus it can be regarded as the *general solution* of the differential equation (1).

Comment: Relation (2) is assumed to be identically satisfied for *all* pairs of variables (x, y). Thus, x and y in (2) are *independent* of each other and so are the differentials dx and dy. On the contrary, the differential equation (1) establishes a connection between the variables x and y, so that the latter becomes a function of the former. This means that (1) is no longer an identity but is satisfied only for certain functions $y=y(x)$, namely, the solutions of the differential equation.

The differential relation (2) is written as

$$M(x,y)\,dx+N(x,y)\,dy=\frac{\partial u}{\partial x}\,dx+\frac{\partial u}{\partial y}\,dy\,.$$

Given that, as remarked above, the differentials dx and dy are independent of each other, the only way to satisfy the above equation is to require that

$$\frac{\partial u}{\partial x}=M(x,y)\ ,\quad \frac{\partial u}{\partial y}=N(x,y) \tag{4}$$

Differentiating the first relation for y and the second one for x, and taking into account that $\partial^2 u/\partial y\partial x=\partial^2 u/\partial x\partial y$ (cf. Sec. 8.1), we find:

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \tag{5}$$

Relation (5) is a *necessary condition* for existence of a solution $u(x, y)$ to the system (4) or, equivalently, to the differential relation (2). If such a solution is found, then by (3) we obtain the general solution of the differential equation (1).

The constant C in the solution (3) is determined by the *initial condition* of the problem. If the specific value $x=x_0$ corresponds to the value $y=y_0$, then $C=C_0=u(x_0, y_0)$. We thus get the *particular solution* $u(x,y)=C_0$.

Example: We consider the differential equation

$$(x+y+1)\,dx+(x-y^2+3)\,dy=0\,,\ \text{ with initial condition }\ y=1\ \text{ for }\ x=0\,.$$

Here, $M=x+y+1$, $N=x-y^2+3$ and $\partial M/\partial y=\partial N/\partial x$ (=1). The system (4) is written

$$\partial u/\partial x=x+y+1,\quad \partial u/\partial y=x-y^2+3\,.$$

The first equation yields

$$u=x^2/2+xy+x+\varphi(y)\,,$$

while by the second one we get

$$\varphi'(y)=-y^2+3\ \Rightarrow\ \varphi(y)=-y^3/3+3y+C_1\,.$$

Thus,

$$u = x^2/2 - y^3/3 + xy + x + 3y + C_1 \,.$$

The general solution (3) is $u(x,y)=C_2$. Putting $C_2 - C_1 \equiv C$, we have:

$$x^2/2 - y^3/3 + xy + x + 3y = C \quad \text{(general solution)} \,.$$

Making the substitutions $x=0$ and $y=1$ (as required by the initial condition) we find $C=8/3$ and

$$x^2/2 - y^3/3 + xy + x + 3y = 8/3 \quad \text{(particular solution)} \,.$$

Exercise 8.3 Show that every *separable* differential equation of the form

$$dy/dx = f(x)\,/\,g(y)$$

is exact.

8.3 Integrating Factor

Assume that the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{1}$$

is not exact; i.e., the left-hand side is not a total differential of some function $u(x,y)$. We say that this equation admits an *integrating factor* $\mu(x,y)$ if there exists a function $\mu(x,y)$ such that the differential equation $\mu(Mdx+Ndy)=0$ *is* exact; that is, the expression $\mu(Mdx+Ndy)$ is a total differential of a function $u(x,y)$:

$$\mu(x,y)\,[M(x,y)\,dx + N(x,y)\,dy] = du(x,y) \,.$$

Then the original equation (1) reduces to the differential relation $du=0 \Rightarrow$

$$u(x,y) = C \tag{2}$$

on the condition that the function $\mu(x,y)$ does not vanish identically when x and y are related by (2). Equation (2) is the general solution of the differential equation (1).

Example: The differential equation $ydx - xdy=0$ is not exact since $M=y$, $N=-x$ and $\partial M/\partial y=1$, $\partial N/\partial x=-1$. However, the equation

$$\frac{1}{y^2}(ydx - xdy) = 0$$

is exact, given that the left-hand side is equal to $d(x/y)$. Thus,

$$d(x/y) = 0 \;\Rightarrow\; y = Cx \,.$$

The solution is acceptable since the integrating factor $\mu=1/y^2$ does not vanish identically for $y=Cx$.

8.4 Line Integrals on the Plane

Consider the xy-plane with coordinates (x, y). Let L be an *oriented curve* (*path*) on the plane, with initial point A and final point B (Fig. 8.1). The curve L may be described by parametric equations of the form

$$\{ x=x(t)\ ,\ \ y=y(t) \} \tag{1}$$

By eliminating t between these equations we obtain a relation of the form $F(x,y)=0$ which, in certain cases, may be written in the form of a function $y=y(x)$.

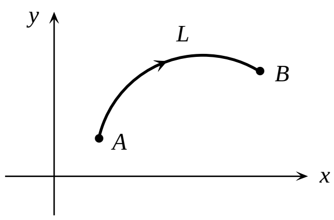


Fig. 8.1. An oriented curve on the xy-plane.

Example: Consider the parametric curve (semicircle) shown in Fig. 8.2:

$$\{ x = R\cos t\ ,\ \ y = R\sin t \}\ \ ,\quad 0 \le t \le \pi\ .$$

The orientation of the curve depends on whether t increases ("counterclockwise") or decreases ("clockwise") between 0 and π. By eliminating t, we get

$$x^2 + y^2 - R^2 = 0 \ \ \Rightarrow\ \ y = (R^2 - x^2)^{1/2}.$$

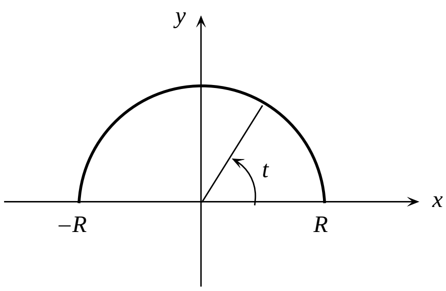


Fig. 8.2. A semicircle.

Given a plane curve L from A to B, as well as two differentiable functions $P(x, y)$ and $Q(x,y)$, we consider an integral of the form

$$I_L = \int_L P(x,y)\,dx + Q(x,y)\,dy \tag{2}$$

Since the *path of integration* consists of the points (x,y) of a certain curve, an integral of the form (2) is called *line integral*. In the parametric form (1) of L we have:

$$dx = (dx/dt)\,dt = x'(t)\,dt\ ,\quad dy = y'(t)\,dt$$

so that

$$I_L = \int_{t_A}^{t_B} \{ P[x(t), y(t)]\,x'(t) + Q[x(t), y(t)]\,y'(t) \}\,dt \tag{3}$$

In the form $y=y(x)$ of L, we write $dy=y'(x)dx$ and

$$I_L = \int_{x_A}^{x_B} \{P[x,y(x)] + Q[x,y(x)]\, y'(x)\}\, dx \tag{4}$$

In general, the value of a line integral I_L *depends on the path* L connecting A and B (not just on the choice of the end points A and B).

Example: We want to compute the line integral (4) for $P(x,y)=Q(x,y)=xy$, along the parabola $y=x^2$ from $x=-1$ to $x=+1$.

Solution: Along the parabola $y=x^2$ we have $P(x,y)=Q(x,y)=xy=x^3$. Moreover, $y'(x)=2x$. From (4) we then get

$$I_L = \int_{-1}^{1} \left(x^3 + 2x^4\right) dx$$

which is easy to compute. (Complete the computation; recall what was said in Sec. 5.3 regarding integrals of even and odd functions.)

For every path L: $A \to B$, we can define the path $-L$: $B \to A$, with *opposite* orientation. From (3) it follows that, if

$$I_L = \int_{t_A}^{t_B} (\cdots)\, dt$$

then

$$I_{-L} = \int_{t_B}^{t_A} (\cdots)\, dt \;.$$

Thus,

$$I_{-L} = -\, I_L \;\;.$$

If the end points A and B of a path coincide, then we have a *closed curve* C and, correspondingly, a *closed line integral* I_C , for which we use the symbol $\oint_C$. We then have:

$$\oint_{-C} (\cdots) = -\oint_C (\cdots)$$

where the orientation of $-C$ is *opposite* to that of C (e.g., if C is counterclockwise on the plane, then $-C$ is clockwise).

Example: The parametric curve

$$\{\, x = R\cos t \;,\;\; y = R\sin t \,\} \;\;,\quad 0 \le t \le 2\pi$$

represents a circle on the plane (Fig. 8.3). If the *counterclockwise* orientation of the circle (where *t increases* from 0 to 2π) corresponds to the curve C, then the *clockwise* orientation (with *t decreasing* from 2π to 0) corresponds to the curve $-C$.

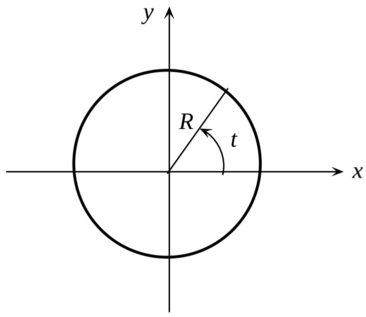


Fig. 8.3. A circle on the *xy*-plane.

Proposition: Let $P(x,y)$ and $Q(x,y)$ be differentiable functions. If

$$\oint_C Pdx + Qdy = 0$$

for *every* closed curve C on the *xy*-plane, then the line integral

$$\int_L Pdx + Qdy$$

is *independent of the path L* connecting any two points A and B on this plane (Fig. 8.4). The converse is also true.

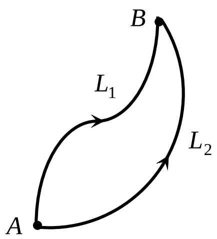


Fig. 8.4. Two paths connecting points *A* and *B* on a plane.

Proof: We consider any two points A and B on the plane, as well as two different paths L_1 and L_2 connecting these points, as seen in the above figure (there is an infinite number of such paths). We form the closed path $C=L_1+(-L_2)$ from A to B through L_1 and back again to A through $-L_2$. We then have:

$$\oint_C Pdx + Qdy = 0 \;\Leftrightarrow\; \int_{L_1} Pdx + Qdy + \int_{-L_2} Pdx + Qdy = 0 \;\Leftrightarrow$$

$$\int_{L_1} Pdx + Qdy - \int_{L_2} Pdx + Qdy = 0 \;\Leftrightarrow\; \int_{L_1} Pdx + Qdy = \int_{L_2} Pdx + Qdy \,.$$

As can be proven [1,2], the path independence of the integral $\int_L Pdx + Qdy$ suggests that the expression $Pdx+Qdy$ is an *exact differential*. That is, there exists a function $u(x,y)$ such that

$$du = Pdx + Qdy \quad \Leftrightarrow \quad \partial u/\partial x = P \,, \quad \partial u/\partial y = Q \tag{5}$$

Moreover, the functions $P(x,y)$ and $Q(x,y)$ satisfy the relation

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad (6)$$

***Exercise 8.*4** Justify the above relation (see Sec. 8.2).

Exercise 8.5 (a) Show that the functions $P(x, y)=y$ and $Q(x, y)=x$ satisfy the condition (6) of path independence. Using (5) and working as in the Example of Sec. 8.2, determine the function $u(x,y)$.
[*Ans.* $u(x,y)= xy+C$]

(b) Repeat the problem for $P(x, y)=x$ and $Q(x, y)=y$.
[*Ans.* $u(x,y)= (x^2+y^2)/2+C$]

References

1. A. F. Bermant, I. G. Aramanovich, *Mathematical Analysis* (Mir Publishers, 1975).
2. C. J. Papachristou, *Aspects of Integrability of Differential Systems and Fields - A Mathematical Primer for Physicists* (Springer, 2019).

CHAPTER 9

COMPLEX NUMBERS

9.1 The Notion of a Complex Number

Consider the equation $x^2+1=0$. Obviously, it cannot be satisfied for any real value of x. We now extend the set of numbers beyond the real numbers by defining the *imaginary unit number* i by

$$i^2 = -1 \quad \text{or, symbolically,} \quad i = \sqrt{-1} \ .$$

Then, the solution of the equation $x^2+1=0$ is $x = \pm i$.

Given the *real* numbers x and y, we define the *complex number*

$$z = x + i\,y \ .$$

This is often represented as an ordered pair:

$$z = x + i\,y \equiv (x, y) \ .$$

The number $x = \text{Re}\, z$ is the *real part* of z, while $y = \text{Im}\, z$ is the *imaginary part* of z. In particular, the value $z = 0$ corresponds to $x = 0$ and $y = 0$. In general, if $y = 0$ then z is a *real* number.

Given a complex number $z = x + i\,y$, the number

$$\bar{z} = x - i\,y \equiv (x, -y)$$

is called the *complex conjugate* of z (the symbol z^* is also used for the complex conjugate). Furthermore, the *real* quantity

$$|z| = (x^2 + y^2)^{1/2}$$

is called the *modulus* (or absolute value) of z. We notice that

$$|z| = |\bar{z}| \ .$$

Example: If $z = 3+2\,i$, then $\bar{z} = 3 - 2\,i$ and $|z| = |\bar{z}| = \sqrt{13}$.

Exercise 9.1 Show that, if $z = \bar{z}$, then z is *real*, and conversely.

Exercise 9.2 Show that, if $z = x + i\,y$, then

$$\text{Re}\, z = x = \frac{z + \bar{z}}{2} \ , \quad \text{Im}\, z = y = \frac{z - \bar{z}}{2i} \ .$$

Consider the complex numbers $z_1 = x_1 + i\,y_1$, $z_2 = x_2 + i\,y_2$. As we can show, their sum and their difference are given by

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)\ ,$$

$$z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)\ .$$

Exercise 9.3 Show that, if $z_1 = z_2$, then $x_1 = x_2$ and $y_1 = y_2$.

Taking into account that $i^2 = -1$, we find the product of z_1 and z_2 to be

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)\ .$$

In particular, for $z_1 = z = x + i\,y$ and $z_2 = \bar{z} = x - i\,y$, we have:

$$z\,\bar{z} = x^2 + y^2 = |\,z\,|^2\,.$$

To evaluate the quotient z_1 / z_2 $(z_2 \neq 0)$ we apply the following trick:

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|\,z_2\,|^2} = \frac{(x_1 + i\,y_1)(x_2 - i\,y_2)}{x_2{}^2 + y_2{}^2} = \frac{x_1 x_2 + y_1 y_2}{x_2{}^2 + y_2{}^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2{}^2 + y_2{}^2}\ .$$

In particular, for $z = x + i\,y$ the *inverse* of z is

$$\frac{1}{z} = \frac{\bar{z}}{z\,\bar{z}} = \frac{\bar{z}}{|\,z\,|^2} = \frac{x - i\,y}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2}\ .$$

Properties:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2\ , \qquad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2\ , \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

$$|\,\bar{z}\,| = |\,z\,|\ , \qquad z\,\bar{z} = |\,z\,|^2\ , \qquad |\,z_1 z_2\,| = |\,z_1\,|\,|z_2\,|$$

$$|\,z^n\,| = |\,z\,|^n\ , \qquad \left|\frac{z_1}{z_2}\right| = \frac{|\,z_1\,|}{|\,z_2\,|}$$

Exercise 9.4 Given the complex numbers $z_1 = 3 - 2\,i$ and $z_2 = -2 + i$, evaluate the quantities $|\,z_1 \pm z_2\,|$, $\bar{z}_1 z_2$ and $\overline{z_1 / z_2}$.

9.2 Polar Form of a Complex Number

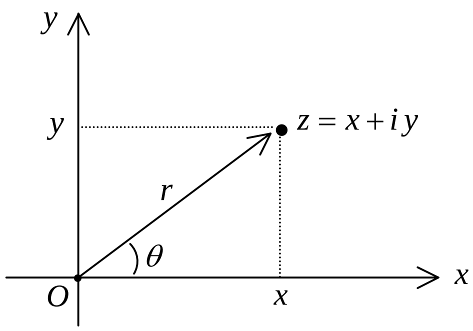


Fig. 9.1. Vector representation of a complex number z.

A complex number $z = x + i\,y \equiv (x, y)$ corresponds to a point of the xy-plane. It may also be represented by a vector joining the origin O of the axes of the complex plane with this point (Fig. 9.1). The quantities x and y are the Cartesian coordinates of the point, or, the orthogonal components of the corresponding vector. We observe that

$$x = r\cos\theta\ , \quad y = r\sin\theta$$

where

$$r = |z| = (x^2 + y^2)^{1/2} \quad \text{and} \quad \tan\theta = \frac{y}{x}\ .$$

Thus, we can write

$$\boxed{z = x + i\,y = r\,(\cos\theta + i\sin\theta)}$$

The above expression represents the *polar form* of z. Note that

$$\bar{z} = r(\cos\theta - i\sin\theta)\,.$$

Let $z_1 = r_1\,(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2\,(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. As can be shown [1],

$$z_1 z_2 = r_1 r_2\,[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]\ ,$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]\ .$$

In particular, the inverse of a complex number $z = r\,(\cos\theta + i\sin\theta)$ is written

$$z^{-1} = \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{1}{r}\,[\cos(-\theta) + i\sin(-\theta)]\ .$$

Exercise 9.5 By using polar forms, show analytically that $z\,z^{-1} = 1$.

9.3 Exponential Form of a Complex Number

We introduce the notation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

(this notation is not arbitrary but has a deeper meaning that reveals itself within the context of the theory of analytic functions; see Chap. 10). Note that

$$e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta\ .$$

Also,

$$|e^{i\theta}| = |e^{-i\theta}| = \cos^2\theta + \sin^2\theta = 1\,.$$

Exercise 9.6 Show that

$$e^{-i\theta} = 1/e^{i\theta} = \overline{e^{i\theta}}\,.$$

Also show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\quad,\quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\ .$$

The complex number $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$, may now be expressed in *exponential form*:

$$\boxed{z = r\,e^{i\theta}}$$

As can be verified,

$$z^{-1} = \frac{1}{z} = \frac{1}{r}\,e^{-i\theta} = \frac{1}{r}\,e^{i(-\theta)}\,,\qquad \bar{z} = r\,e^{-i\theta}$$

$$z_1 z_2 = r_1 r_2\,e^{i(\theta_1+\theta_2)}\,,\qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}\,e^{i(\theta_1-\theta_2)}$$

where $z_1 = r_1\,e^{i\theta_1}$, $z_2 = r_2\,e^{i\theta_2}$.

Example: The complex number $z = \sqrt{2} - i\sqrt{2}$, with $|z| = r = 2$, is written

$$z = 2\left(\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right) = 2\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right] = 2\,e^{i(-\pi/4)} = 2\,e^{-i\pi/4}\,.$$

9.4 Powers and Roots of Complex Numbers

Let $z = r\,(\cos\theta + i\,\sin\theta) = r\,e^{i\theta}$ be a complex number, where $r = |z|$. As can be proven,

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta) \qquad (n = 0, \pm 1, \pm 2, \cdots) \ .$$

In particular, for $z = \cos\theta + i\,\sin\theta = e^{i\theta}$ $(r=1)$ we find the *de Moivre's formula*

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta) \ .$$

Note also that, for $z \neq 0$, we have that $z^0 = 1$ and $z^{-n} = 1/z^n$.

Given a complex number $z = r\,(\cos\theta + i\,\sin\theta)$, where $r = |z|$, an *nth root of* z is any complex number c satisfying the equation $c^n = z$. We write $c = \sqrt[n]{z}$. The nth root of a complex number admits n different values given by the formula [1]

$$c_k = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0, 1, 2, \cdots, (n-1) \ .$$

Example: Let $z = 1$. We seek the 4th roots of unity, i.e., the complex numbers c satisfying the equation $c^4 = 1$. We write

$$z = 1\,(\cos 0 + i\,\sin 0) \qquad (\text{that is, } r = 1\,,\ \theta = 0)\ .$$

Then,

$$c_k = \cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4} = \cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2} \ , \quad k = 0, 1, 2, 3 \ .$$

We find: $c_0 = 1$, $c_1 = i$, $c_2 = -1$, $c_3 = -i$.

Example: Let $z = i$. We seek the square roots of i, that is, the complex numbers c satisfying the equation $c^2 = i$. We have:

$$z = 1\,[\cos(\pi/2) + i\,\sin(\pi/2)] \qquad (\text{that is, } r = 1\,,\ \theta = \pi/2)\ ;$$

$$c_k = \cos\frac{(\pi/2) + 2k\pi}{2} + i\sin\frac{(\pi/2) + 2k\pi}{2} \ , \quad k = 0, 1 \ ;$$

$$c_0 = \cos(\pi/4) + i\sin(\pi/4) = \frac{\sqrt{2}}{2}\,(1+i) \ ,$$

$$c_1 = \cos(5\pi/4) + i\sin(5\pi/4) = -\frac{\sqrt{2}}{2}\,(1+i) \ .$$

Reference

1. R. V. Churchill, J. W. Brown, *Complex Variables and Applications*, 5th Edition (McGraw-Hill, 1990).

CHAPTER 10

INTRODUCTION TO COMPLEX ANALYSIS

10.1 Analytic Functions and the Cauchy-Riemann Relations

Complex analysis (the theory of complex functions and their differentiation and integration) is a subject too deep to treat in a short chapter. We will thus only give some elements of this subject, "borrowing" some material from a previous book by this author [1].

We consider complex functions of the form

$$w = f(z) = u(x, y) + i\,v(x, y) \tag{1}$$

where $z=x+iy \equiv (x, y)$ is a point on the complex plane, and where u and v are real functions of x and y. Let Δz be a change of z and let $\Delta w=f(z+\Delta z) - f(z)$ be the corresponding change of the value of $f(z)$. We say that the function (1) is *differentiable* at point z if we can write

$$\frac{\Delta w}{\Delta z} = f'(z) + \varepsilon(z, \Delta z) \quad \text{with} \quad \lim_{\Delta z \to 0} \varepsilon(z, \Delta z) = 0 \tag{2}$$

Then, the function

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{3}$$

is the *derivative* of $f(z)$ at point z. Evidently, in order for $f(z)$ to be differentiable at z, this function must be *defined* at that point. We also note that a function differentiable at a point z_0 is necessarily *continuous* at z_0 (the converse is not always true) [2-4]. That is, $\lim\limits_{z \to z_0} f(z) = f(z_0)$ (assuming that the limit exists).

A function $f(z)$ differentiable at every point of a domain G of the complex plane is said to be *analytic* (or *holomorphic*) in the domain G. The criterion for analyticity is the validity of a pair of partial differential equations (PDEs) called the *Cauchy-Riemann relations*.

Theorem: Consider a complex function $f(z)$ of the form (1), continuous at every point $z \equiv (x, y)$ of a domain G of the complex plane. The real functions $u(x,y)$ and $v(x,y)$ are differentiable at every point of G and, moreover, their partial derivatives with respect to x and y are continuous functions in G. Then, the function $f(z)$ is analytic in the domain G if and only if the following system of PDEs is satisfied [2-4]:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4}$$

It is convenient as well as economical, with regard to notation, to denote partial derivatives by using subscripts:

$$\frac{\partial\phi}{\partial x}\equiv\phi_x \ , \quad \frac{\partial\phi}{\partial y}\equiv\phi_y \ , \quad \frac{\partial^2\phi}{\partial x^2}\equiv\phi_{xx} \ , \quad \frac{\partial^2\phi}{\partial y^2}\equiv\phi_{yy} \ , \quad \frac{\partial^2\phi}{\partial x\partial y}\equiv\phi_{xy} \ , \quad \text{etc.}$$

The Cauchy-Riemann relations (4) then read

$$u_x = v_y \ , \qquad u_y = -v_x \tag{4′}$$

The derivative of the function (1) may now be expressed in the following alternate forms:

$$f'(z) = u_x + i v_x = v_y - i u_y = u_x - i u_y = v_y + i v_x \tag{5}$$

Comments:

1. Relations (4) allow us to find v when we know u, and vice versa. Let us put $u_x=P$, $u_y=Q$, so that $\{v_x=-Q\ ,\ v_y=P\}$. The *integrability* (*compatibility*) *condition* of this system for solution for v, for a given u, is found by equating the "mixed" partial derivatives of v (see Sec. 8.1): $(v_y)_x = (v_x)_y \Rightarrow$

$$\partial P/\partial x = -\,\partial Q/\partial y \ \Rightarrow\ u_{xx}+u_{yy}= 0\ .$$

Similarly, the integrability condition of system (4) for solution for u, for a given v, is $v_{xx}+v_{yy}= 0$. We notice that both the real and the imaginary part of an analytic function are *harmonic functions*, i.e., they satisfy the *Laplace equation*

$$w_{xx} + w_{yy} = 0 \tag{6}$$

Harmonic functions related to each other by means of the Cauchy-Riemann relations (4) are called *conjugate harmonic*.

2. Let $\bar{z} = x - i y$ be the complex conjugate of $z=x+iy$. Then,

$$x = (z+\bar{z})/2 \ , \quad y = (z-\bar{z})/2i \tag{7}$$

By using relations (7) we can express $u(x,y)$ and $v(x,y)$, thus also the sum $w=u+iv$, as functions of z and $\bar{z}$. The real Cauchy-Riemann relations (4), then, are rewritten in the form of a single complex equation [2-4]

$$\partial w/\partial\bar{z} = 0 \tag{8}$$

One way to interpret this result is the following: The analytic function (1) is *literally* a function of the complex variable $z=x+iy$, not just some complex function of two real variables x and y!

Examples:

1. We seek an analytic function of the form (1), with $v(x,y)=xy$. Note first that v satisfies the PDE (6): $v_{xx}+v_{yy}=0$ (harmonic function). Thus, the integrability condition of the system (4) for solution for u is satisfied. The system is written

$$\partial u/\partial x=x\,, \quad \partial u/\partial y=-y\,.$$

The first relation yields

$$u = x^2/2 + \varphi(y)\,.$$

From the second one we then get

$$\varphi'(y) = -y \;\;\Rightarrow\;\; \varphi(y) = -y^2/2 + C$$

so that

$$u = (x^2-y^2)/2 + C\,.$$

Putting $C=0$, we finally have

$$w = u + iv = (x^2-y^2)/2 + ixy\,.$$

Exercise 10.1 Show that $u_{xx}+u_{yy}=0$; i.e., $u(x,y)$ is a harmonic function.

Exercise 10.2 Using relations (7), show that $w=f(z)=z^2/2$, thus verifying condition (8).

2. Consider the function $w=f(z)=|z|^2$ defined on the entire complex plane. Here, $u(x,y)=x^2+y^2$, $v(x,y)=0$. As is easy to verify, the Cauchy-Riemann relations (4) are not satisfied anywhere on the plane, except at the single point $z=0$ where $(x,y)\equiv(0,0)$. Alternatively, we may write $w=z\bar{z}$, so that $\partial w/\partial\bar{z}=z\neq 0$ (except at $z=0$). We conclude that the given function is not analytic on the complex plane.

3. In Chapter 2 we learned that the simple exponential function $f(x)=e^x\equiv\exp x$ is the only real function that equals its own derivative, i.e., satisfies the differential equation $f'(x)=f(x)$. By extension, the function $f(z)=e^z\equiv\exp z$ is defined as the complex function that satisfies the differential relation $f'(z)=f(z)$. As can be proven [2-4], this function is given by the formula

$$f(z)=e^z=e^{x+iy}=e^x(\cos y+i\sin y) \tag{9}$$

This function is analytic on the entire complex plane. Notice that for $x=0$ we have the important formula

$$e^{iy}=\cos y+i\sin y$$

which we used in Sec. 9.3 to express the exponential form of a complex number.

Exercise 10.3 For the exponential function (9), identify the real functions $u(x,y)$ and $v(x,y)$ [see Eq. (1)] and show that the Cauchy-Riemann relations (4) are satisfied.

10.2 Integrals of Complex Functions

Let L be an oriented curve on the complex plane (Fig. 10.1), the points of which plane are represented as $z=x+iy \equiv (x, y)$. The points z of L are determined by some parametric relation of the form

$$z = \lambda(t) = x(t) + i\,y(t) \quad , \qquad \alpha \le t \le \beta \tag{1}$$

As t increases from α to β, the curve L is traced from A to B, while the opposite curve $-L$ is traced from B to A with t decreasing from β to α (see Sec. 8.4).

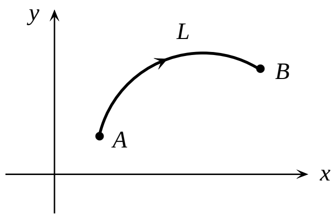


Fig. 10.1. An oriented curve on the complex plane.

We now consider integrals of the form $\int_L f(z)\,dz$, where $f(z)$ is a complex function. We write $dz=\lambda'(t)\,dt$, so that

$$\int_L f(z)\,dz = \int_\alpha^\beta f[\lambda(t)]\lambda'(t)\,dt \tag{2}$$

Also,

$$\int_{-L} f(z)\,dz \;=\; \int_\beta^\alpha (\cdots)\,dt \;=\; -\int_\alpha^\beta (\cdots)\,dt \quad \Rightarrow$$

$$\int_{-L} f(z)\,dz \;=\; -\int_L f(z)\,dz \tag{3}$$

A *closed* curve C will be conventionally regarded as *positively* oriented if it is traced *counterclockwise*. Then, the opposite curve $-C$ will be *negatively* oriented and will be traced *clockwise*. Moreover,

$$\oint_{-C} f(z)\,dz \;=\; -\oint_C f(z)\,dz \tag{4}$$

Examples:

1. We want to evaluate the integral

$$I = \oint_{|z-a|=\rho} \frac{dz}{z-a} \;,$$

where the circle $|z-a|=\rho$ is traced (a) counterclockwise, (b) clockwise.

(*a*) The circle $|z-a|=\rho$ is described parametrically by the relation

$$z=a+\rho e^{it} , \ 0\leq t \leq 2\pi .$$

Then,

$$dz = (a+\rho e^{it})' dt = i\rho e^{it} dt .$$

Integrating from 0 to 2π (for *counterclockwise* tracing) we have:

$$I = \int_0^{2\pi} \frac{i\rho e^{it} dt}{\rho e^{it}} = i\int_0^{2\pi} dt \quad \Rightarrow \quad \oint_{|z-a|=\rho} \frac{dz}{z-a} = 2\pi i \ .$$

(*b*) For *clockwise* tracing of the circle $|z-a|=\rho$, we write, again,

$$z=a+\rho e^{it} \ (0\leq t \leq 2\pi) .$$

This time, however, we integrate from 2π to 0. Then,

$$I = i\int_{2\pi}^{0} dt = -2\pi i .$$

Alternatively, we write

$$z=a+\rho e^{-it} \ (0\leq t \leq 2\pi)$$

and integrate from 0 to 2π, arriving at the same result.

2. Consider the integral

$$I = \oint_{|z-a|=\rho} \frac{dz}{(z-a)^2} ,$$

where the circle $|z-a|=\rho$ is traced *counterclockwise*. We write

$$z=a+\rho e^{it} \ (0\leq t \leq 2\pi)$$

so that

$$I = \int_0^{2\pi} \frac{i\rho e^{it} dt}{\rho^2 e^{2it}} = \frac{i}{\rho}\int_0^{2\pi} e^{-it} dt = 0 \ .$$

In general, for $k = 0, \pm1, \pm2, \ldots$ and for a *positively* (*counterclockwise*) oriented circle $|z-a|=\rho$, one can show that

$$\oint_{|z-a|=\rho} \frac{dz}{(z-a)^k} = \begin{cases} 2\pi i \ , & \text{if} \quad k=1 \\ 0 \ , & \text{if} \quad k\neq 1 \end{cases} \tag{5}$$

10.3 The Cauchy-Goursat Theorem

We now state an important theorem concerning analytic functions [2-4].

Theorem: Assume that the function $f(z)=u(x,y)+i\,v(x,y)$ is analytic at *all* points of a domain G of the complex plane, bounded by a closed curve C (among other things, this means that $f(z)$ is defined *everywhere* in the interior of C).[1] Then,

$$\oint_C f(z)\,dz = 0 \tag{1}$$

Corollary: The line integral of the analytic function $f(z)$ is independent of the path connecting any two points A and B of the domain G.

Proof: As in Sec. 8.4, we consider two paths L_1 and L_2 (Fig. 10.2) and we form the closed path $C=L_1+(-L_2)$. By (1) we then have:

$$\oint_C f(z)\,dz = \int_{L_1} f(z)\,dz + \int_{-L_2} f(z)\,dz = 0 \iff$$

$$\int_{L_1} f(z)\,dz - \int_{L_2} f(z)\,dz = 0 \iff \int_{L_1} f(z)\,dz = \int_{L_2} f(z)\,dz \,.$$

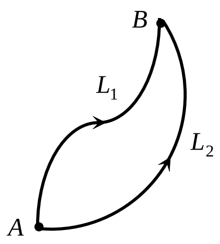


Fig. 10.2. Two paths connecting points A and B on the complex plane.

10.4 Indefinite Integral of an Analytic Function

Let z_0 and z be two points of a domain G of the complex plane. We regard z_0 as constant while z is assumed to be variable. According to the Cauchy-Goursat theorem, the line integral from z_0 to z, of a function $f(z)$ analytic in G, depends only on the two limit points and is independent of the curved path connecting them. Hence, such an integral may be denoted by

$$\int_{z_0}^{z} f(z')\,dz'$$

or, for simplicity,

$$\int_{z_0}^{z} f(z)\,dz \,.$$

For variable upper limit z, this integral is a function of its upper limit. We write

$$\int_{z_0}^{z} f(z)\,dz = I(z) \tag{1}$$

[1] Note, for example, that the function $f(z)=1/z^k$ is not defined for $z=0$. Thus (1) is not valid for $k=1$ if C encircles the origin of the complex plane. [It *is* valid, however, for $k\neq 1$; see Eq. (5) of Sec. 10.2.]

As can be shown [2], $I(z)$ is an analytic function. Moreover, it is an *antiderivative* of $f(z)$; that is, $I'(z)=f(z)$. Analytically,

$$I'(z)=\frac{d}{dz}\int_{z_0}^{z} f(z)\,dz = f(z) \tag{2}$$

Any antiderivative $F(z)$ of $f(z)$ [$F'(z)=f(z)$] is equal to $F(z)=I(z)+C$, where $C=F(z_0)$ is a constant [notice that $I(z_0)=0$]. We observe that $I(z)=F(z)-F(z_0) \Rightarrow$

$$\int_{z_0}^{z} f(z)\,dz = F(z)-F(z_0) \tag{3}$$

In general, for given z_1, z_2 and for an *arbitrary* antiderivative $F(z)$ of $f(z)$, we may write

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2)-F(z_1) \tag{4}$$

Now, if we also allow the lower limit z_0 of the integral in relation (1) to vary, then this relation yields an *infinite set of antiderivatives* of $f(z)$, which set represents the *indefinite integral* of $f(z)$ and is denoted by $\int f(z)\,dz$. If $F(z)$ is any antiderivative of $f(z)$, then, by relation (3) and by putting $-F(z_0)=C$,

$$\int f(z)\,dz = \{F(z)+C \;/\; F'(z)=f(z),\; C=const.\}\,.$$

To simplify our notation, we write

$$\int f(z)\,dz = F(z)+C \tag{5}$$

where the right-hand side represents an *infinite set* of functions, not just any specific antiderivative of $f(z)$!

Examples:

1. The function $f(z)=z^2$ is analytic on the entire complex plane and one of its antiderivatives is $F(z)=z^3/3$. Thus,

$$\int z^2 dz = \frac{z^3}{3}+C \quad \text{and} \quad \int_{-1}^{i} z^2 dz = \frac{1}{3}(1-i)\ .$$

2. The function $f(z)=1/z^2$ is differentiable everywhere except at the origin O of the complex plane, where $z=0$. An antiderivative, for $z\neq 0$, is $F(z)=-1/z$. Hence,

$$\int \frac{dz}{z^2} = -\frac{1}{z}+C \quad \text{and} \quad \int_{z_1}^{z_2} \frac{dz}{z^2} = \frac{1}{z_1}-\frac{1}{z_2}$$

where the path connecting the points $z_1\neq 0$ and $z_2\neq 0$ does not pass through O.

References

1. C. J. Papachristou, *Aspects of Integrability of Differential Systems and Fields - A Mathematical Primer for Physicists* (Springer, 2019).
2. A. I. Markushevich, *The Theory of Analytic Functions: A Brief Course* (Mir Publishers, 1983).
3. L. V. Ahlfors, *Complex Analysis*, 3rd Edition (McGraw-Hill, 1979).
4. R. V. Churchill, J. W. Brown, *Complex Variables and Applications*, 5th Edition (McGraw-Hill, 1990).

APPENDIX

Trigonometric Formulas

$$\sin^2 A+\cos^2 A=1 \quad ; \quad \tan x=\frac{\sin x}{\cos x} \quad ; \quad \cot x=\frac{\cos x}{\sin x}=\frac{1}{\tan x}$$

$$\cos^2 x=\frac{1}{1+\tan^2 x} \quad ; \quad \sin^2 x=\frac{1}{1+\cot^2 x}=\frac{\tan^2 x}{1+\tan^2 x}$$

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$

$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B} \quad , \quad \cot(A\pm B)=\frac{\cot A\cot B\mp 1}{\cot B\pm\cot A}$$

$$\sin 2A=2\sin A\cos A$$

$$\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A} \quad , \quad \cot 2A=\frac{\cot^2 A-1}{2\cot A}$$

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A-\sin B=2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$$

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A-\cos B=2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$$

$$\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\cos A\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)]$$

$$\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$\sin(-A)=-\sin A \quad , \quad \cos(-A)=\cos A$$

$$\tan(-A)=-\tan A \quad , \quad \cot(-A)=-\cot A$$

$$\sin(\frac{\pi}{2}\pm A)=\cos A \quad , \quad \cos(\frac{\pi}{2}\pm A)=\mp\sin A$$

$$\sin(\pi\pm A)=\mp\sin A \quad , \quad \cos(\pi\pm A)=-\cos A$$

	sin	cos	tan	cot
0	0	1	0	∞
$\pi/6 = 30^{o}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
$\pi/4 = 45^{o}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
$\pi/3 = 60^{o}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
$\pi/2 = 90^{o}$	1	0	∞	0
$\pi = 180^{o}$	0	–1	0	∞

Basic Trigonometric Equations

$$\sin x = \sin\alpha \;\Rightarrow\; \begin{cases} x=\alpha+2k\pi \\ x=(2k+1)\pi-\alpha \end{cases} \qquad (k = 0, \pm1, \pm2, \cdots)$$

$$\cos x = \cos\alpha \;\Rightarrow\; \begin{cases} x=\alpha+2k\pi \\ x=2k\pi-\alpha \end{cases} \qquad (k = 0, \pm1, \pm2, \cdots)$$

$$\tan x = \tan\alpha \;\Rightarrow\; x = \alpha + k\pi \qquad (k = 0, \pm1, \pm2, \cdots)$$

$$\cot x = \cot\alpha \;\Rightarrow\; x = \alpha + k\pi \qquad (k = 0, \pm1, \pm2, \cdots)$$

$$\sin x = -\sin\alpha \;\Rightarrow\; \begin{cases} x=2k\pi-\alpha \\ x=\alpha+(2k+1)\pi \end{cases} \qquad (k = 0, \pm1, \pm2, \cdots)$$

$$\cos x = -\cos\alpha \;\Rightarrow\; \begin{cases} x=(2k+1)\pi-\alpha \\ x=\alpha+(2k+1)\pi \end{cases} \qquad (k = 0, \pm1, \pm2, \cdots)$$

Power Formulas

$(a\pm b)^2 = a^2 \pm 2ab + b^2$

$(a\pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

$a^2 - b^2 = (a+b)(a-b)$

$a^3 \pm b^3 = (a\pm b)(a^2 \mp ab + b^2)$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n \quad (n=1,2,3,\cdots)$$

Quadratic Equation: $ax^2 + bx + c = 0$

Call $D = b^2 - 4ac$ (discriminant)

Roots: $x = \dfrac{-b\pm\sqrt{D}}{2a}$

Roots are real and distinct if $D>0$; real and equal if $D=0$; complex conjugates if $D<0$.

Hyperbolic Functions

$\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{1}{\coth x}$

$\cosh^2 x - \sinh^2 x = 1$

$\cosh(-x) = \cosh x$, $\sinh(-x) = -\sinh x$

$(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$

Properties of Inequalities

$a < b \;\; and \;\; b < c \;\Rightarrow\; a < c$

$a \geq b \;\; and \;\; b \geq a \;\Rightarrow\; a = b$

$a < b \;\Rightarrow\; -a > -b$

$0 < a < b \;\Rightarrow\; \frac{1}{a} > \frac{1}{b}$

$a < b \;\; and \;\; c \leq d \;\Rightarrow\; a + c < b + d$

$0 < a < b \;\; and \;\; 0 < c \leq d \;\Rightarrow\; ac < bd$

$0 < a < 1 \;\Rightarrow\; a > a^2 > a^3 > \cdots \;, \quad a^n < 1 \,, \quad \sqrt[n]{a} < 1$

$a > 1 \;\Rightarrow\; a < a^2 < a^3 < \cdots \;, \quad a^n > 1 \,, \quad \sqrt[n]{a} > 1$

$0 < a < b \;\Rightarrow\; a^n < b^n \,, \quad \sqrt[n]{a} < \sqrt[n]{b}$

Properties of Proportions

Assume that $\frac{\alpha}{\beta} = \frac{\gamma}{\delta} = \kappa$. Then,

$\alpha\delta = \beta\gamma$, $\qquad \frac{\alpha \pm \gamma}{\beta \pm \delta} = \kappa$

$\frac{\alpha \pm \beta}{\beta} = \frac{\gamma \pm \delta}{\delta}$, $\qquad \frac{\alpha}{\beta \pm \alpha} = \frac{\gamma}{\delta \pm \gamma}$

Properties of Absolute Values of Real Numbers

$$
\begin{aligned}
|a| &= a, \quad if \quad a \geq 0 \\
&= -a, \quad if \quad a < 0
\end{aligned}
$$

$$|a| \geq 0$$

$$|-a| = |a|$$

$$|a|^2 = a^2$$

$$\sqrt{a^2} = |a|$$

$$|x| \leq \varepsilon \quad \Leftrightarrow \quad -\varepsilon \leq x \leq \varepsilon \quad (\varepsilon > 0)$$

$$|x| \geq a > 0 \quad \Leftrightarrow \quad x \geq a \quad or \quad x \leq -a$$

$$\big||a| - |b|\big| \leq |a \pm b| \leq |a| + |b|$$

$$|a \cdot b| = |a||b|$$

$$|a^k| = |a|^k \quad (k \in Z)$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad (b \neq 0)$$

Properties of Powers and Logarithms

$$x^0 = 1 \qquad (x \neq 0)$$

$$x^\alpha x^\beta = x^{\alpha+\beta}$$

$$\frac{x^\alpha}{x^\beta} = x^{\alpha-\beta}$$

$$\frac{1}{x^\alpha} = x^{-\alpha}$$

$$\left(x^\alpha\right)^\beta = x^{\alpha\beta}$$

$$(xy)^\alpha = x^\alpha y^\alpha \quad ; \qquad \left(\frac{x}{y}\right)^\alpha = \frac{x^\alpha}{y^\alpha}$$

$$\ln 1 = 0$$

$$\ln\left(e^\alpha\right) = \alpha \quad (\alpha \in \mathbb{R}) \quad , \qquad e^{\ln\alpha} = \alpha \quad (\alpha \in \mathbb{R}^+)$$

$$\ln(\alpha\beta) = \ln\alpha + \ln\beta$$

$$\ln\left(\frac{\alpha}{\beta}\right) = \ln\alpha - \ln\beta = -\ln\left(\frac{\beta}{\alpha}\right)$$

$$\ln\left(\frac{1}{\alpha}\right) = -\ln\alpha$$

$$\ln\left(\alpha^k\right) = k\ln\alpha \quad (k \in \mathbb{R})$$

Continuity and Differentiability

Consider a function $y=f(x)$ defined in an open interval containing the point $x=x_0$. At this point the value of f is $y_0=f(x_0)$. Assume further that the limit of $f(x)$ for $x \to x_0$ exists and is equal to $f(x_0)$:

$$\lim_{x\to x_0} f(x) = f(x_0) \qquad (1)$$

Then $f(x)$ is said to be *continuous* at $x=x_0$.

Put $x-x_0 = \Delta x$ and $f(x) - f(x_0) = y - y_0 = \Delta y$. If $x \to x_0$ then $\Delta x \to 0$. From (1) we have:

$$\lim_{x\to x_0} f(x) - f(x_0) = \lim_{x\to x_0}\left[f(x) - f(x_0)\right] = 0 \;\; \Rightarrow$$

$$\lim_{\Delta x\to 0} \Delta y = 0 \quad \Leftrightarrow \quad \Delta y \to 0 \;\; when \;\; \Delta x \to 0 \qquad (2)$$

Theorem: Let $y=f(x)$ be defined in an open interval containing x_0. If the derivative $f'(x_0)$ exists, then $f(x)$ is continuous at $x=x_0$. (The converse is not necessarily true.)

Proof: We must show that $\Delta y \to 0$ when $\Delta x \to 0$, where $\Delta x = x-x_0$ and where

$$\Delta y = y - y_0 = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0) .$$

We have that $f'(x_0) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$. But,

$$\lim_{\Delta x\to 0} \Delta y = \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\, \Delta x \right) = \left(\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}\right)\left(\lim_{\Delta x\to 0} \Delta x\right) = f'(x_0) \cdot 0 = 0$$

given that, by assumption, the derivative $f'(x_0)$ exists (in particular, it assumes a finite value). Thus condition (2) is satisfied, i.e., $f(x)$ is continuous at $x=x_0$.

ANSWERS TO SELECTED EXERCISES

1.1 (1) $D=R$ (2) $D=[-1, 1]$ (3) $D=(-\infty, -1)\cup(1, +\infty)$ (4) $D=[-1/3, 1]$
(5) $D=(-5, 2)$ (6) $D=(1, +\infty)$ (7) $D=R-\{k\pi/2+\pi/4\}$ (8) $D=R-\{3k\pi+3\pi/2\}$

1.3 (1) 0 (2) -2 (3) 1

1.5 (1) even (2) odd (3) neither (4) even (5) odd
(6) odd (7) even (8) odd (9) even (10) odd

1.8 (1) not periodic (2) $a=12\pi$ (3) $a=\pi$ (4) $a=\pi/\lambda$

2.1 (1) $y'=1/4\sqrt{x}+1/\sqrt{x^3}$ (2) $y'=(x\cos x-\sin x)/x^2$
(3) $y'=(3\sqrt{x}+2\sqrt{x^3})e^x-3(1-\ln x)/x^2$

2.2 (1) $y'=2xy[\sin(3x^2+1)]^{-2/3}\cos(3x^2+1)$
(2) $y'=-2x^5(x^6+1)^{-2/3}\sin\left(2\sqrt[3]{x^6+1}\right)$ (3) $y'=3\sin 2x\sin 4x/\cos^2(\sin^3 2x)$
(4) $y'=4x^3/(x^4+1)\ln(x^4+1)$ (5) $y'=x\left(\ln\sqrt{x^2+1}\right)^{-1/2}/2(x^2+1)$
(6) $y'=y[1+\ln(x+1)]$ (7) $y'=xy(1+2\ln x)$
(8) $y'=y[\cos x\cot x-\sin x\ln(\sin x)]$ (9) $y'=1/x$

3.2 (1) 3 (2) 0 (3) $1/\sqrt{e}$ (4) $1/e$

4.1 (1) $-\frac{2}{x}-3\ln x+\sqrt{x}+C$ (2) $3x-2\ln x+\frac{8}{3}\sqrt{x^3}+C$

4.4 (1) $\ln(\ln x)+C$ (2) $-\frac{1}{2}e^{1/x^2}+C$ (3) $\frac{1}{4}[\ln(x^2+1)]^2+C$
(4) $\ln(\tan x)+C$ (5) $2\sin(e^{\sqrt{x}})+C$ (6) $\frac{1}{\sqrt{5}}\arctan(x/\sqrt{5})+C$
(7) $\frac{1}{3}\arctan(\frac{x}{3}-1)+C$

4.5 (1) $(x^2-2x+2)e^x+C$ (2) $\frac{e^x}{2}(\sin x-\cos x)+C$
(3) $\frac{1}{2}(x-\sin x\cos x)+C=\frac{1}{2}(x-\frac{\sin 2x}{2})+C$
(4) $\frac{1}{2}(x+\sin x\cos x)+C=\frac{1}{2}(x+\frac{\sin 2x}{2})+C$

4.6 (1) $e^{\sqrt{x}}(\cos\sqrt{x}+\sin\sqrt{x})+C$ (2) $\sin^2 x[\ln(\sin x)-1/2]+C$

5.2 (1) $(\ln 5)/2$ (2) $\sqrt{e}-1$ (3) $2/\pi$

SELECTED BIBLIOGRAPHY

D. D. Berkey, *Calculus*, 2nd Edition (Saunders College, 1988).

A. F. Bermant, I. G. Aramanovich, *Mathematical Analysis* (Mir Publishers, 1975).

M. L. Boas, *Mathematical Methods in the Physical Sciences*, 3rd Edition (Wiley, 2006).

L. Elsgolts, *Differential Equations and the Calculus of Variations* (Mir Publishers, 1977).

M. D. Greenberg, *Advanced Engineering Mathematics*, 2nd Edition (Prentice-Hall, 1998).

O. V. Manturov, N. M. Matveev, *A Course of Higher Mathematics* (Mir Publishers, 1989).

J. Mathews, R. L. Walker, *Mathematical Methods of Physics*, 2nd Edition (W. A. Benjamin, 1970).

C. J. Papachristou, *Aspects of Integrability of Differential Systems and Fields - A Mathematical Primer for Physicists* (Springer, 2019).

M. Spivak, *Calculus*, 3rd Edition (Cambridge University Press, 1994).

G. B. Thomas, R. L. Finney, *Calculus and Analytic Geometry,* 9th Edition (Addison-Wesley, 1998).

R. Wrede, M. R. Spiegel, *Advanced Calculus*, 2nd Edition (Schaum's Outline Series, McGraw-Hill, 2002).

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