Transformation Lie groups and operator representations

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A note on continuous groups of transformations on linear spaces and manifolds and the operator representations of transformation Lie groups and algebras.

1. Lie groups and Lie algebras: An overview

We review some basic definitions concerning Lie groups and Lie algebras.

A *group* is a set $G=\{a,b,c,...\}$ equipped with an internal "multiplication" operation with the following properties:

1. Closure: $ab \in G,\ \forall\, a, b \in G$.
2. Associativity: $a(bc) = (ab)c$.
3. Identity element: $\exists\, e \in G$: $ae = ea,\ \forall\, a \in G$.
4. Inverse element: $\forall\, a \in G,\ \exists\, a^{-1} \in G$: $aa^{-1} = a^{-1}a = e$.

A group is *abelian* (or commutative) if $ab = ba,\ \forall\, a, b \in G$.

A *subgroup* of G is a subset $H \subseteq G$ that is itself a group under the group operation of G. Obviously, H must contain the identity element e of G as well as the inverse of any element of H.

A map $\varphi : G \to G'$ from a group G to a group G' is called a *homomorphism* if it preserves group multiplication. That is, for any a, $b \in G$, the images $\varphi(a) \in G'$ and $\varphi(b) \in G'$ satisfy the relation

$$\varphi(a)\,\varphi(b) = \varphi(ab) .$$

If the homomorphism φ is 1-1, it is called an *isomorphism*. An isomorphic relation of G with a group of matrices or operators is called a matrix or operator *representation* of G, accordingly.

A real *Lie algebra* $\mathcal{L}$ of dimension n is an n-dimensional real vector space equipped with an internal *Lie bracket* operation [,] that satisfies the following properties:

1. Closure: $[a,b] \in \mathcal{L},\ \forall\, a, b \in \mathcal{L}$.
2. Linearity: $[\kappa a + \lambda b,\, c] = \kappa [a,c] + \lambda [b,c]$ $(\kappa, \lambda \in R)$.
3. Antisymmetry: $[a,b] = -[b,a]$. Corollary: $[a,a] = 0$.
4. Jacobi identity: $\big[a,[b,c]\big] + \big[b,[c,a]\big] + \big[c,[a,b]\big] = 0$.

A Lie algebra is *abelian* (or commutative) if $[a,b] = 0$, $\forall\, a, b \in \mathcal{L}$.

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A *subalgebra* S of $\mathcal{L}$ is a subspace of $\mathcal{L}$ that itself is a Lie algebra. The algebra S is an *invariant subalgebra* or *ideal* of $\mathcal{L}$ if $[a,b]\in S,\ \forall\, a\in S,\ b\in\mathcal{L}$. A Lie algebra $\mathcal{L}$ is said to be *simple* if it contains no ideals other than itself; $\mathcal{L}$ is *semisimple* if it contains no *abelian* ideals.

Examples of Lie algebras:

1. The algebra of ($m\times m$) matrices, with $[A,B]=AB-BA$ (*commutator*). Diagonal matrices constitute an abelian subalgebra of this algebra.

2. The algebra of all vectors in 3-dimensional space, with $[\vec{V},\vec{W}]=\vec{V}\times\vec{W}$ (vector product). Vectors parallel to a given axis form an abelian subalgebra of this algebra.

A map $\psi: \mathcal{L}\rightarrow\mathcal{L}'$ from a Lie algebra $\mathcal{L}$ to a Lie algebra $\mathcal{L}'$ is a *homomorphism* if it satisfies the following properties:

$$\psi(\kappa a+\lambda b) = \kappa\,\psi(a)+\lambda\,\psi(b) \quad (\kappa,\lambda\in R)\ ;$$
$$\psi([a,b]) = [\psi(a),\psi(b)]\ .$$

If the map ψ is 1-1, it is called an *isomorphism*. Isomorphic Lie algebras $\mathcal{L}$ and $\mathcal{L}'$ have equal dimensions [1]: $dim\mathcal{L}=dim\mathcal{L}'$.

Let $\{\tau_i\ /\ i=1,2,\dots,n\}$ be a basis of an n-dimensional Lie algebra $\mathcal{L}$. Since the Lie bracket of any two basis elements τ_i and τ_j is an element of $\mathcal{L}$, it must be a linear combination of the $\{\tau_k\}$. That is,

$$[\tau_i\,,\tau_j]=C_{ij}^k\,\tau_k \tag{1}$$

(sum on k from 1 to n). By the antisymmetry of the Lie bracket, $C_{ij}^k=-C_{ji}^k$. The real constants C_{ij}^k are called *structure constants* of the Lie algebra $\mathcal{L}$.

Proposition 1: Let $\psi: \mathcal{L}\rightarrow\mathcal{L}'$ be a Lie algebra isomorphism. If $\{\tau_k\}$ $(k=1,2,\dots,n)$ is a basis of $\mathcal{L}$, then $\{\psi(\tau_k)\}$ is a basis of $\mathcal{L}'$.

Proof: Being a basis of $\mathcal{L}$, the $\{\tau_k\}$ are linearly independent; hence no linear combination of them can be zero (unless, of course, all coefficients are trivially zero). Now, by the properties of ψ, a linear combination of the $\{\tau_k\}$ is mapped onto a linear combination of the $\{\psi(\tau_k)\}$ with the same coefficients. This means that the latter combination cannot vanish, since it can only be zero if the former one is zero as well; that is, if all coefficients in the combination are zero. We conclude that the $\{\psi(\tau_k)\}$ are linearly independent and may serve as a basis for $\mathcal{L}'$.

Proposition 2: Isomorphic Lie algebras share common structure constants.

Proof: Let $\psi: \mathcal{L}\rightarrow\mathcal{L}'$ be a Lie algebra isomorphism and let τ_i, τ_j be any two basis elements of $\mathcal{L}$. Then, $\psi(\tau_i)$ and $\psi(\tau_j)$ are basis elements of $\mathcal{L}'$. By (1) and by the properties of ψ,

$$\psi([\tau_i,\tau_j])=\psi(C_{ij}^k\,\tau_k)\;\Rightarrow\;[\psi(\tau_i),\psi(\tau_j)]=C_{ij}^k\,\psi(\tau_k)\,;\;\text{q.e.d.}$$

Roughly speaking, a *Lie group* is a group G whose elements depend on a number of parameters that can be varied in a continuous way. The *dimension* n of G is the number of real parameters parametrizing the elements of G. We assume that $dimG=n$ and we let $\{\lambda^1,\lambda^2,\dots,\lambda^n\}$ be the set of n parameters of G. We arrange the parameterization of G so that the identity element of G corresponds to $\lambda^k=0$ for all $k=1,2,...,n$.

An important class of Lie groups consists of groups of $(m\times m)$ matrices parametrized by n parameters λ^k ($k=1,2,...,n$). Since an $(m\times m)$ matrix produces a *linear transformation* on an m-dimensional Euclidean space, matrix groups are called *linear groups*.

Lie groups are closely related to Lie algebras. Let G be an n-dimensional Lie group of $(m\times m)$ matrices $A(\lambda^1,\lambda^2,\dots,\lambda^n)\equiv A(\lambda)$ (where by λ we collectively denote the set of the n parameters λ^k). We define the n $(m\times m)$ matrices τ_k by

$$\tau_k=\frac{\partial A(\lambda)}{\partial\lambda^k}\Big|_{\lambda^1=\lambda^2=\cdots=\lambda^n=0} \tag{2}$$

or, in terms of matrix elements,

$$(\tau_k)_{pq}=\frac{\partial A_{pq}}{\partial\lambda^k}\Big|_{\lambda^1=\lambda^2=\cdots=\lambda^n=0}$$

($k=1,2,...,n$; $p,q=1,2,...,m$). The n matrices τ_k are called *infinitesimal operators* (or generators) of the Lie group G and form the basis of an n-dimensional real Lie algebra $\mathcal{L}$ [1]. Thus $[\tau_i,\tau_j]=C_{ij}^k\,\tau_k$, where the C_{ij}^k are real constants. A general element a of $\mathcal{L}$ is written as a linear combination of the τ_k: $a=\xi^k\tau_k$ (sum on k), for real coefficients ξ^k. [Note carefully that the matrix elements $(\tau_k)_{pq}$ *themselves* are *not* required to be real numbers!]

Now, let $a=\lambda^k\tau_k$ be the general element of $\mathcal{L}$. The general element $A(\lambda)$ of the Lie group G parametrized by the λ^k can then be written as [1,2]

$$A(\lambda)=e^a=\exp(\lambda^k\tau_k) \tag{3}$$

where e^a is the matrix exponential function

$$e^a\equiv\exp a=\sum_{l=0}^{\infty}\frac{a^l}{l!}=1+a+\frac{a^2}{2}+\cdots$$

For infinitesimal values of the parameters λ^k we may use the approximate expression

$$e^a\simeq 1+a$$

so that

$$A(\lambda)\simeq 1+\lambda^k\tau_k\;.$$

The simplest example of a Lie group is a one-parameter continuous group, such as the group $SO(2)$ of rotations on a plane. A rotation of a vector by an angle λ is represented by the (2×2) orthogonal matrix

$$A(\lambda) = \begin{bmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{bmatrix} \quad (\lambda \in R) .$$

(Notice that $A^t A=1$ and det$A$=1.) Then

$$\frac{dA}{d\lambda} = \begin{bmatrix} -\sin\lambda & -\cos\lambda \\ \cos\lambda & -\sin\lambda \end{bmatrix}$$

and, by Eq. (2), the single basis element τ of the associated Lie algebra is

$$\tau = \frac{dA}{d\lambda}\Big|_{\lambda=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

According to (3), $A(\lambda)=e^{\lambda\tau}$ and, for infinitesimal λ, $A(\lambda) \simeq 1+\lambda\tau$. Indeed, by setting $\sin\lambda=\lambda$ and $\cos\lambda=1$, we have:

$$A(\lambda) \simeq \begin{bmatrix} 1 & -\lambda \\ \lambda & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 1 + \lambda\tau .$$

Another single-parameter Lie group is the unitary group $U(1)$ with elements $\{e^{i\lambda}\}$ ($\lambda \in R$), which may be regarded as (1×1) matrices. Consider the map φ: $U(1) \to SO(2)$ defined by

$$\varphi\left(e^{i\lambda}\right) = \begin{bmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{bmatrix} .$$

This map is a homomorphism, since

$$\begin{aligned} \varphi\left(e^{i\lambda} \cdot e^{i\lambda'}\right) = \varphi\left(e^{i(\lambda+\lambda')}\right) &= \begin{bmatrix} \cos(\lambda+\lambda') & -\sin(\lambda+\lambda') \\ \sin(\lambda+\lambda') & \cos(\lambda+\lambda') \end{bmatrix} \\ &= \begin{bmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{bmatrix} \begin{bmatrix} \cos\lambda' & -\sin\lambda' \\ \sin\lambda' & \cos\lambda' \end{bmatrix} \\ &= \varphi\left(e^{i\lambda}\right) \cdot \varphi\left(e^{i\lambda'}\right) . \end{aligned}$$

Moreover, it can be shown [1] that the map φ is 1-1. Therefore, φ is a Lie-group isomorphism.

We finally remark that isomorphic Lie groups have isomorphic Lie algebras [1]. More generally, under certain restrictions, homomorphic Lie groups may have isomorphic Lie algebras, as the case is with the groups $SU(2)$ and $SO(3)$.

2. Group operators for a linear Lie group

Let $(x^1, \ldots, x^m)$ be a system of coordinates on R^m. Consider an n-dimensional Lie group G represented by $(m\times m)$ matrices g the elements of which depend on n real parameters $(a^1, \ldots, a^n)$. We call $\vec{x}$ the column vector with components $(x^1, \ldots, x^m)$. The action of G on this vector is expressed as

$$\vec{x} \to g\vec{x}\ , \quad g \in G \tag{4}$$

In effect, Eq. (4) describes a linear coordinate transformation[2] on R^m.

Let $\{L_1, \ldots, L_n\}$ be a basis of the Lie algebra of G, where the $\{L_\gamma\}$ are $(m\times m)$ matrices.[3] Then there exist real structure constants $C^\gamma_{\alpha\beta}$ such that the following commutation relations are satisfied:

$$[L_\alpha\, , L_\beta] = C^\gamma_{\alpha\beta} L_\gamma \quad (\text{sum on } \gamma) \tag{5}$$

An element $g\in G$ can then be put in the form $g=\exp(a^\lambda L_\lambda)$ [1,2] so that (4) is written: $\vec{x} \to \exp(a^\lambda L_\lambda)\,\vec{x}$. For infinitesimal values δa^λ of the group parameters,

$$\exp(\delta a^\lambda L_\lambda) \simeq 1 + \delta a^\lambda L_\lambda$$

so that $\vec{x} \to (1+\delta a^\lambda L_\lambda)\,\vec{x} \equiv \vec{x} + \delta\vec{x}$, where

$$\delta\vec{x} = \delta a^\lambda L_\lambda \vec{x} \;\;\Leftrightarrow\;\; \delta x^i = \delta a^\lambda (L_\lambda)^i{}_k\, x^k \tag{6}$$

The expression $g=\exp(a^\lambda L_\lambda)$ is a representation of G in terms of linear coordinate transformations (4) on R^m. We now seek a different realization of G in terms of transformations of functions $F(\vec{x})$, $\vec{x} \in R^m$. We define the operators

$$T(g):\; F \to T(g)F, \quad g\in G$$

by

$$[T(g)\,F](\vec{x}) = F\left(g^{-1}\vec{x}\right) \tag{7}$$

Proposition 1: The operators $T(g)$ constitute an operator representation of G.

Proof: Let $g_1, g_2 \in G$. Then, for an arbitrary function F on R^m,

$$\begin{aligned}[T(g_1 g_2)F](\vec{x}) &= F\left(g_2^{-1} g_1^{-1}\vec{x}\right) = [T(g_2)F]\left(g_1^{-1}\vec{x}\right) = \left\{T(g_1)[T(g_2)F]\right\}(\vec{x}) \\ &\equiv \left\{[T(g_1)T(g_2)]F\right\}(\vec{x}) \;\;\Rightarrow\end{aligned}$$

$$T(g_1 g_2) = T(g_1)\,T(g_2)\ , \quad \text{q.e.d.}$$

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[2] For definiteness we regard this as an *active* transformation from a point $x\in R^m$ with coordinates x^k to a point x' with coordinates $x^{k\prime} = (gx)^k$.
[3] Greek indices run from 1 to n while Latin indices run from 1 to m. The summation convention will be used throughout.

For $g=\exp(a^\lambda L_\lambda) \simeq 1+\delta a^\lambda L_\lambda$ we have that $g^{-1} \simeq 1-\delta a^\lambda L_\lambda$, and so (7) yields, by using Eq. (A.1) in the Appendix:

$$[T(g)F](\vec{x}) \simeq F\left(\vec{x}-\delta a^\lambda L_\lambda \vec{x}\right) \simeq F(\vec{x}) - \delta a^\lambda L_\lambda \vec{x}\cdot\vec{\nabla}F(\vec{x})$$
$$= \left(1-\delta a^\lambda L_\lambda \vec{x}\cdot\vec{\nabla}\right)F(\vec{x}) \;\; \Rightarrow$$

$$T(g) \simeq 1-\delta a^\lambda L_\lambda \vec{x}\cdot\vec{\nabla} \equiv 1+\delta a^\lambda P_\lambda \tag{8}$$

where $P_\lambda = -L_\lambda \vec{x}\cdot\vec{\nabla} = -(L_\lambda \vec{x})^i \dfrac{\partial}{\partial x^i} \;\; \Rightarrow$

$$P_\lambda = -(L_\lambda)^i{}_k \, x^k \frac{\partial}{\partial x^i} \equiv -(L_\lambda)^i{}_k \, x^k \partial_i \tag{9}$$

where we have introduced the notation $\partial_i \equiv \partial/\partial x^i$. For finite values of the group parameters a^λ, Eq. (8) generalizes to $T(g)=\exp(a^\lambda P_\lambda)$ [3,4].

Proposition 2: The operators $\{P_\lambda\}$ are the basis of a Lie algebra isomorphic to the Lie algebra of the matrices $\{L_\gamma\}$. Thus, if the commutation relations (5) are valid, then also

$$[P_\alpha \, , \, P_\beta] = C^\gamma_{\alpha\beta} P_\gamma \tag{10}$$

Proof: Consider the linear mapping

$$\Psi: \; L \rightarrow P = \Psi(L) = -L^i{}_k \, x^k \partial_i \tag{11}$$

where the matrix L is an element of the Lie algebra of G. Let L_1 , L_2 be two such elements. Then,

$$P_1 = \Psi(L_1) = -(L_1)^i{}_k \, x^k \partial_i \;\; , \quad P_2 = \Psi(L_2) = -(L_2)^i{}_k \, x^k \partial_i \;\; .$$

We have:

$$\Psi([L_1 \, , L_2]) = \Psi(L_1L_2 - L_2L_1) = \Psi(L_1L_2) - \Psi(L_2L_1) \quad \text{(since } \Psi \text{ is linear)}$$
$$= -(L_1L_2)^i{}_k \, x^k \partial_i + (L_2L_1)^i{}_k \, x^k \partial_i$$
$$= -(L_1)^i{}_j (L_2)^j{}_k \, x^k \partial_i + (L_2)^i{}_j (L_1)^j{}_k \, x^k \partial_i \, .$$

On the other hand,

$$[\Psi(L_1) \, , \Psi(L_2)] = [P_1 \, , P_2] = P_1P_2 - P_2P_1$$
$$= (L_1)^i{}_j \, x^j \partial_i [(L_2)^k{}_l \, x^l \partial_k] - (L_2)^k{}_l \, x^l \partial_k [(L_1)^i{}_j \, x^j \partial_i] \, .$$

After a lengthy but straightforward calculation, and by canceling out second-order derivatives, we find:

$$[\Psi(L_1) \, , \Psi(L_2)] = -(L_1)^i{}_j (L_2)^j{}_k \, x^k \partial_i + (L_2)^i{}_j (L_1)^j{}_k \, x^k \partial_i \, .$$

We thus conclude that

$$\Psi([L_1, L_2]) = [\Psi(L_1), \Psi(L_2)]$$

which is what we needed to prove. Moreover,

$$[P_\alpha\,,\,P_\beta] = [\Psi(L_\alpha)\,,\,\Psi(L_\beta)] = \Psi\left([L_\alpha\,,\,L_\beta]\right) = \Psi\left(C^\gamma_{\alpha\beta} L_\gamma\right)$$
$$= C^\gamma_{\alpha\beta}\,\Psi(L_\gamma) = C^\gamma_{\alpha\beta} P_\gamma$$

which verifies (10).

Example: Let $G=SO(3)$, the group of (3×3) real orthogonal matrices with unit determinant. It is a 3-parameter Lie group [1,5] and thus the associated Lie algebra *so*(3) is 3-dimensional. The basis of *so*(3) consists of the (3×3) antisymmetric matrices

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with commutation relations

$$[\,L_i\,,\,L_j\,] = \varepsilon_{ijk} L_k \quad \text{(sum on } k\text{)}$$

where ε_{ijk} is antisymmetric in all pairs of indices, with $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$. [We use Latin instead of Greek indices for the basis elements of *so*(3) since the number of these elements matches the dimensions of R^3, on which space both the *SO*(3) and *so*(3) matrices act.] We notice that

$$(L_i)^j{}_k = -\,\varepsilon_{ijk}\ .$$

The operator representation of the basis of *so*(3) is, according to (9),

$$P_i = -\,(L_i)^j{}_k\, x^k \partial_j = \varepsilon_{ijk}\, x^k \partial_j$$

or, analytically,

$$P_1 = x^3 \partial_2 - x^2 \partial_3\ ,\quad P_2 = x^1 \partial_3 - x^3 \partial_1\ ,\quad P_3 = x^2 \partial_1 - x^1 \partial_2\ .$$

The reader may check that $[P_i\,,\,P_j] = \varepsilon_{ijk} P_k$.

3. Group operators for general coordinate transformations

The previous results are valid for *linear* (matrix) groups, in which case $g\vec{x}$ represents the action of an $(m\times m)$ matrix on a vector of R^m. More generally, consider an m-dimensional manifold M with coordinates $(x^1, \dots, x^m)$ and let G be an n-dimensional local Lie group of coordinate transformations on M (see [6] for rigorous definitions and examples). The elements g of G depend on n real parameters $(a^1, \dots, a^n)$. We call $x \equiv (x^1, \dots, x^m)$ a point on M and we denote by gx a (possibly nonlinear) coordinate transformation on this manifold. To the first order in the group parameters a^λ, i.e., for infinitesimal δa^λ, such a transformation is approximately linear in the δa^λ. We write:

$$(gx)^i \simeq x^i + \delta x^i \quad \text{where} \quad \delta x^i = \delta a^\lambda U_\lambda^i(x^k) \tag{12}$$

$(i = 1, \dots, m;\ \lambda = 1, \dots, n)$.

Let $F(x)$ be an arbitrary function on M. As before, we define the operators

$$T(g):\ F \to T(g)F, \quad g \in G$$

by

$$[T(g)F](x) = F\left(g^{-1}x\right) \tag{13}$$

Again, the $T(g)$ constitute an operator representation of G:

$$T(g_1 g_2) = T(g_1)\,T(g_2)\ .$$

[Careful: $g_1 g_2$ is no longer a matrix product but a succession of coordinate transformations! It is still true, however, that $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$.]

Given that, by (12),

$$(gx)^i \simeq x^i + \delta a^\lambda U_\lambda^i(x^k)$$

we have that

$$(g^{-1}x)^i \simeq x^i - \delta a^\lambda U_\lambda^i(x^k)\ .$$

Let us justify this statement:

$$\begin{aligned}(g^{-1}gx)^i &\simeq (gx)^i - \delta a^\lambda U_\lambda^i\left((gx)^k\right) \\ &\simeq x^i + \delta a^\lambda U_\lambda^i(x^k) - \delta a^\lambda U_\lambda^i\left(x^k + \delta a^\rho U_\rho^k\right)\end{aligned}$$

By using Eq. (A.1) in the Appendix we have that, to the first order in the δa^λ,

$$\begin{aligned}\delta a^\lambda U_\lambda^i\left(x^k + \delta a^\rho U_\rho^k\right) &\simeq \delta a^\lambda \left[U_\lambda^i(x^k) + \delta a^\rho U_\rho^j \partial_j U_\lambda^i(x^k)\right] \\ &\simeq \delta a^\lambda U_\lambda^i(x^k)\end{aligned}$$

Thus, finally, $(g^{-1}gx)^i = x^i \ \Leftrightarrow\ g^{-1}gx \equiv$ identity transformation.

By using (A.1) once more, the infinitesimal version of (13) is written:

$$[T(g)F](x) \simeq F\left(x^i - \delta a^\lambda U_\lambda^i\right) \simeq F(x) - \delta a^\lambda U_\lambda^i \partial_i F(x)$$
$$= \left(1 - \delta a^\lambda U_\lambda^i \partial_i\right) F(x) \;\Rightarrow$$

$$T(g) \simeq 1 - \delta a^\lambda U_\lambda^i \partial_i \equiv 1 + \delta a^\lambda P_\lambda \tag{14}$$

where

$$P_\lambda = -U_\lambda^i(x^k)\,\partial_i \tag{15}$$

It can be proven [3] that the operators P_λ ($\lambda = 1, \dots, n$) form the basis of an n-dimensional Lie algebra:

$$[P_\alpha\,, P_\beta] = C_{\alpha\beta}^\gamma P_\gamma \tag{16}$$

Let us see what this implies: Let

$$P_\alpha = -U_\alpha^i(x^k)\,\partial_i\ , \quad P_\beta = -U_\beta^j(x^k)\,\partial_j\ .$$

Then,

$$[P_\alpha\,, P_\beta] = \left(U_\alpha^i \partial_i U_\beta^j - U_\beta^i \partial_i U_\alpha^j\right)\partial_j\,.$$

A set of real constants $C_{\alpha\beta}^\gamma$ must then exist such that

$$U_\alpha^i \partial_i U_\beta^j - U_\beta^i \partial_i U_\alpha^j = -C_{\alpha\beta}^\gamma U_\gamma^j \tag{17}$$

Then,

$$[P_\alpha\,, P_\beta] = -C_{\alpha\beta}^\gamma U_\gamma^j\,\partial_j = C_{\alpha\beta}^\gamma P_\gamma\ .$$

Relations (17) are conditions for closure, under the Lie bracket, of the set of operators spanned by the basis $\{P_\lambda\}$; that is, conditions in order that this set constitute a Lie algebra.

Appendix: Multidimensional Taylor expansion

The Taylor series expansion of a function $f(x)$ about a point x can be written as

$$f(x+h)=\sum_{n=0}^{\infty}\frac{1}{n!}h^n\left(\frac{d}{dx}\right)^n f(x)=f(x)+h\frac{df(x)}{dx}+\cdots$$

We write:

$$f(x+h)=\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(h\frac{d}{dx}\right)^n\right]f(x)=\exp\left(h\frac{d}{dx}\right)f(x)\ .$$

For infinitesimal $h\equiv\delta x$ we may use the approximate expression

$$f(x+\delta x)\simeq f(x)+\delta x\frac{df(x)}{dx}\ .$$

More generally, consider a function $\Phi(x^1,x^2,\cdots)\equiv\Phi(\vec{r})$. Let $\vec{a}\equiv(a^1,a^2,\cdots)$ be a constant vector. Then,

$$\Phi(\vec{r}+\vec{a})=\sum_{n=0}^{\infty}\frac{1}{n!}(\vec{a}\cdot\vec{\nabla})^n\Phi(\vec{r})=\Phi(\vec{r})+\vec{a}\cdot\vec{\nabla}\Phi(\vec{r})+\cdots$$

where $\vec{\nabla}\Phi\equiv(\partial\Phi/\partial x^1,\partial\Phi/\partial x^2,\cdots)$. We write:

$$\Phi(\vec{r}+\vec{a})=\left[\sum_{n=0}^{\infty}\frac{1}{n!}(\vec{a}\cdot\vec{\nabla})^n\right]\Phi(\vec{r})=\exp(\vec{a}\cdot\vec{\nabla})\Phi(\vec{r})\ .$$

For infinitesimal $\vec{a}\equiv\delta\vec{r}$,

$$\begin{gathered}\Phi(\vec{r}+\delta\vec{r})\simeq\Phi(\vec{r})+\delta\vec{r}\cdot\vec{\nabla}\Phi(\vec{r})\equiv\Phi(\vec{r})+\delta\Phi\quad\text{where}\\ \delta\Phi=\delta\vec{r}\cdot\vec{\nabla}\Phi(\vec{r})=\delta x^k\,\frac{\partial\Phi(\vec{r})}{\partial x^k}\quad(\text{sum on }k)\end{gathered}\qquad\text{(A.1)}$$

Indeed, notice that, infinitesimally,

$$\delta\Phi\simeq d\Phi=\frac{\partial\Phi}{\partial x^k}dx^k\quad\text{where}\quad dx^k\equiv\delta x^k\ .$$

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