Oscillatory motion on a roller-coaster track

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Problem

A body of mass m is moving back and forth on a U-shaped, frictionless roller-coaster track on the vertical xy-plane, where the x-axis is horizontal while the y-axis is vertical (Fig. 1). The shape of the track, which is symmetric with respect to the y-axis, is described mathematically by an equation of the form $y=f(x)$, where $f(x)$ is an *even* function and where $f(0)=0$. (*a*) Find the differential equation describing the position of m on the track as a function of time. (*b*) Propose a solution to this equation in integral form. (*c*) Determine the period of the oscillatory motion, given the total mechanical energy E of m (equivalently, the maximum height h reached by the body).

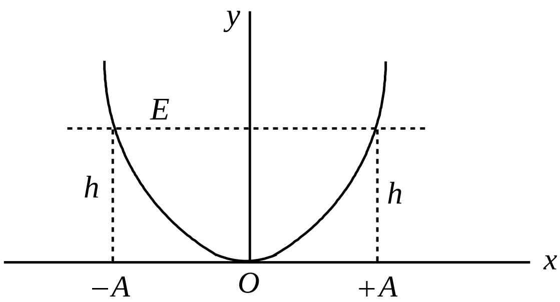

Fig. 1

Solution

Let us first take a look at the physics of the problem. The body m is sliding without friction on a roller-coaster track described by the equation $y=f(x)$, moving back and forth between two extreme points at height h above the x-axis (Fig. 1). The projections of these points on this axis are $-A$ and $+A$. The body is subject to the gravitational force mg and the normal force from the track. The latter force produces no work, hence does not affect the conservation of mechanical energy (see, e.g., Sec. 4.5 of [1]). The gravitational potential energy of m is $U(y)=mgy$. Along the track, where $y=f(x)$, the values of U may be expressed in terms of x:

$$U(x) = mg\, f(x) \qquad (1)$$

Let E be the total mechanical energy of m. Since E is constant along the path, its value will be equal to the value of the potential energy at the extreme positions corresponding to $x=-A$ and $x=+A$ (at which positions the kinetic energy of m vanishes). That is,

$$E = U(\pm A) = mg\, f(\pm A) = mgh \qquad (2)$$

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The kinetic energy of the body is

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right)$$

(dots indicate differentiation with respect to *t*) where, for $y=f(x)$,

$$\dot{y} = \frac{d}{dt} f(x) = \frac{d f(x)}{dx} \frac{dx}{dt} = \dot{x} f'(x) \tag{3}$$

Hence,

$$T = \frac{1}{2} m \dot{x}^2 \left\{ 1 + [f'(x)]^2 \right\} \tag{4}$$

The total mechanical energy $E=T+U$ is constant along the path. By (1), (2) and (4) we have:

$$\frac{1}{2} m \dot{x}^2 \left\{ 1 + [f'(x)]^2 \right\} + mg f(x) = mgh \tag{5}$$

The position of *m* on the track is specified by a single coordinate *x*, which plays the role of a generalized coordinate in the sense of Lagrangian dynamics. The Lagrangian function is

$$L(x, \dot{x}) = T - U = \frac{1}{2} m \dot{x}^2 \left\{ 1 + [f'(x)]^2 \right\} - mg f(x) \tag{6}$$

The Lagrange equation for $x(t)$ is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \tag{7}$$

We note that the time-derivative of *any* function of *x* is defined by the rule used in (3) for $f(x)$. With this in mind, it is a somewhat long but straightforward exercise to show that (6) and (7) yield the differential equation

$$\ddot{x} \left\{ 1 + [f'(x)]^2 \right\} + \dot{x}^2 f'(x) f''(x) + g f'(x) = 0 \tag{8}$$

Presumably, the first-order differential equation (5), expressing conservation of mechanical energy, is a *first integral* of the second-order differential equation (8). [In general, a first integral of a differential equation is a lower-order differential equation (or an algebraic relation, in the case of a first-order equation) that gives us the information that some mathematical quantity retains a constant value as a consequence of the original differential equation. See, e.g., Chaps. 3 and 4 of [2].]

To prove the validity of the above statement, we need to integrate (8) once in order to derive (5). It is easier, however, to work in reverse order. We thus take the time-derivative of (5), keeping the rule (3) in mind. Not surprisingly, the result is again the differential equation (8) (show this)!

The equation of motion of m on the track is a function $x(t)$ that satisfies the differential equation (8). In principle, this second-order equation has “already” been integrated once to obtain the first-order equation (5) [which is a first integral of (8), expressing conservation of mechanical energy]. From (5) we have:

$$\dot{x}^2 = \frac{2g[h - f(x)]}{1 + [f'(x)]^2} \ .$$

This yields a first-order differential equation for $x(t)$:

$$\frac{dx}{dt} = \pm \left\{ \frac{2g[h - f(x)]}{1 + [f'(x)]^2} \right\}^{1/2} \equiv \pm \Lambda(x;h) \tag{9}$$

By assuming the initial condition $x=x_0$ for $t=t_0$, the differential equation (9) is integrated to give

$$\int_{x_0}^{x} \frac{\pm\, dx}{\Lambda(x;h)} = t - t_0 \tag{10}$$

where the plus sign is chosen for motion in the positive direction ($x>x_0$), while the minus sign applies to motion in the negative direction ($x<x_0$). This formally solves the problem of determining the position of m on the track as a function of time.

The period P of the oscillatory motion of m is the time it takes for a complete journey from the extreme position with $x= -A$ to the extreme position with $x= +A$ and back to the original position $x= -A$. To find P we use (10) with the appropriate sign for each direction of motion:

$$P = \int_{-A}^{A} \frac{dx}{\Lambda(x;h)} + \int_{A}^{-A} \frac{-dx}{\Lambda(x;h)} = 2\int_{-A}^{A} \frac{dx}{\Lambda(x;h)} \ .$$

We observe that P depends on the maximum height h, thus on the total energy E of the body (notice that both the integrand *and* the limits of integration depend on h). However, P is independent of the mass of the body, as expected for a motion governed by the sole action of gravity.

References

[1] C. J. Papachristou, *Introduction to Mechanics of Particles and Systems* (Springer, 2020)

(see http://metapublishing.org/index.php/MP/catalog/book/68)

[2] C. J. Papachristou, *Aspects of Integrability of Differential Systems and Fields: A Mathematical Primer for Physicists* (Springer, 2019)

(see https://arxiv.org/abs/1511.01788)