Understanding the “twin paradox”

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It is shown that, among all possible motions in spacetime, inertial motions are the ones exhibiting maximum proper time. By taking this into account, the so-called twin paradox can be resolved.

1. Introduction

The “twin paradox” is a standard paradigm in university-physics textbooks discussing Special Relativity (relativity in flat spacetime); see, e.g., [1-4]. It is certainly the most remarkable example of asymmetry between inertial and non-inertial frames of reference.

The story is well known: The brother stays at home while his twin sister goes on a round trip at very high speed. When she returns home she finds that her brother has grown older than her. Now, the brother can understand why this happened: she has been moving relative to him and hence her clock has been running slower than his. But, from the point of view of his sister, it was *he* that was moving relative to her. Why then isn’t she the older one?

The key to resolve the paradox is to realize that the symmetry (reciprocity) of the time dilation effect exists only between *inertial* observers. And, while the brother *is* such an observer, his sister is *not* as she has experienced acceleration in order to perform her round trip. Hence only the brother is in a position to properly analyze the situation.

The resolution of the twin paradox involves the concept of *proper time*, a Lorentz-invariant quantity representing time measured by a clock along its own worldline (spacetime trajectory). As will be shown, among all possible motions between two fixed points (events) in spacetime, inertial motions are the ones exhibiting maximum proper time. It is thus the stationary brother’s clock that will record the longest time.

So, let us begin by defining proper time.

2. Proper time and time dilation

We consider 4-dimensional flat spacetime. Let $x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z)$ be the spacetime coordinates of an inertial frame of reference S used by an inertial observer (where c is the speed of light in empty space) and let $x^{\mu\prime} \equiv (ct', x', y', z')$ be the coordinates of another frame S' used by a different inertial observer (we recall that the speed of light is a frame-independent quantity). According to Special Relativity [5] the two spacetime coordinate systems are related by the linear *Lorentz transformation*

$$x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu \qquad (1)$$

(the familiar summation convention for repeated up and down indices is assumed) where the constant (4×4) matrix $\Lambda \equiv [\Lambda^\mu{}_\nu]$ $(\mu, \nu = 0,1,2,3)$ satisfies the equation

$$\Lambda^t g \Lambda = g \quad \Leftrightarrow \quad \Lambda^\mu{}_\lambda \, g_{\mu\nu} \, \Lambda^\nu{}_\rho = g_{\lambda\rho} \qquad (2)$$

with

$$g \equiv [g_{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = diag\,(1,-1,-1,-1) \qquad (3)$$

Moreover, Λ is required to obey the constraints

$$\det\Lambda = +1\ , \quad \Lambda^0{}_0 \geq 1 \qquad (4)$$

In group-theoretical terms, Λ belongs to the *restricted Lorentz group* $SO(3,1)\uparrow$.

By differentiating (1) we get the infinitesimal Lorentz transformation

$$dx^{\mu\,\prime} = \Lambda^\mu{}_\nu\, dx^\nu \qquad (5)$$

(since $\Lambda^\mu{}_\nu$ is constant). We define the *spacetime interval*

$$\begin{aligned} ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\ &= c^2 dt^2 - dx^2 - dy^2 - dz^2 \end{aligned} \qquad (6)$$

As can be shown, ds^2 is a *Lorentz scalar,* i.e., a quantity invariant under the transformation (1). This means that

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{\mu\nu}\, dx^{\mu\,\prime} dx^{\nu\,\prime} \qquad (7)$$

and, explicitly,

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 (dt')^2 - (dx')^2 - (dy')^2 - (dz')^2\ .$$

Note that ds^2 may be positive, negative or zero. In particular, the Lorentz invariance of the relation $ds^2 = 0$, valid along the worldline (spacetime trajectory) of a light ray, is equivalent to the invariance of the speed c of light upon passing from one inertial reference frame to another.

If $ds^2 > 0$ (*timelike interval*) then $ds = (ds^2)^{1/2}$ may be an element of spacetime distance along the worldline of a massive particle, as viewed by an inertial observer using a frame S with coordinates x^μ or (x, y, z, t). The worldline need not be straight, which means that the particle may execute *accelerated* motion relative to the frame S. We define the Lorentz scalar $d\tau$ (*proper-time interval*) by

$$d\tau = ds/c \quad \Leftrightarrow \quad ds = c\, d\tau \qquad (8)$$

Proper time represents time as measured by a clock *moving with the particle* (thus moving along the particle's worldline and being at rest relative to the particle). Let us see why.

According to S, the speed of the particle and the clock moving with it is

$$u = dl/dt = (dx^2 + dy^2 + dz^2)^{1/2} / dt$$

where dl is the length of an infinitesimal displacement along the spatial trajectory of the particle. From (6) we have that $ds^2 = c^2 dt^2 - dl^2$ and, given that $dl = u\,dt$,

$$ds^2 = (c^2 - u^2)\, dt^2 \;\Rightarrow\; ds = (c^2 - u^2)^{1/2}\, dt\,.$$

By (8) we then have:

$$d\tau = \left(1 - \frac{u^2}{c^2}\right)^{1/2} dt \;\Leftrightarrow\; dt = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} d\tau \equiv \gamma(u)\, d\tau \tag{9}$$

where $\gamma(u) = (1 - u^2/c^2)^{-1/2}$.

Relation (9) is valid for a frame S with respect to which the particle and the associated clock move with speed u. Since $d\tau$ is a Lorentz scalar, its value is independent of the particular frame S. Consider now a *local* inertial frame S' relative to which the particle and the clock are *momentarily* at rest (obviously, an infinite number of such frames are needed, one for each momentary position of the clock). Then, $u'=0$, $\gamma(u')=1$ and, by (9), $dt' = \gamma(u')\, d\tau = d\tau$. In conclusion:

> *In the rest frame of the particle (and thus of the clock) the proper-time interval $d\tau$ is literally a time interval, as measured by the clock. Proper time is thus actual time measured along the particles own spacetime trajectory (worldline).*

For any other frame S, relative to which $u \neq 0$, we have that $\gamma(u) > 1$ and so $dt > d\tau$, hence $dt > dt'$. This expresses the familiar relativistic effect of *time dilation*. Note carefully that, while the *time* interval dt depends on the particular frame S, the *proper*-time interval $d\tau$ is, by its very definition (8), a frame-independent quantity! However, $d\tau$ is a genuine time interval in the rest frame of the particle, whereas it is a *spacetime* interval in any other frame relative to which the particle is in motion.

3. Time measurements along worldlines; the maximum proper time

To simplify our subsequent analysis, we consider two-dimensional spacetime and an inertial observer using a frame S with coordinates (x, t) (thus we assume that $y=z=0$). The observer observes two clocks (1) and (2) moving along his x-axis and draws their worldlines as shown in Fig. 1.

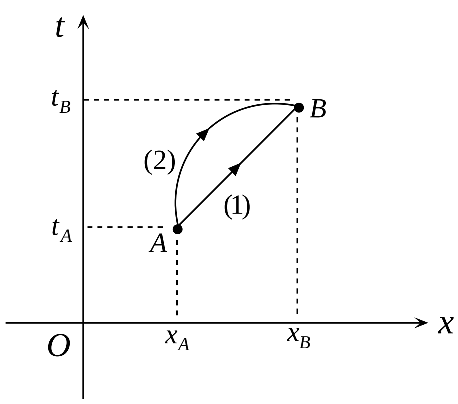

Fig. 1

The clocks start moving together at time t_A , from the same point x_A of the x-axis, and meet again at time t_B at point x_B . Thus the worldlines of both clocks connect the spacetime point (*event*) A with the spacetime point B. Clock (1) moves *inertially* since its worldline is straight, while the motion of clock (2) is *accelerated* in view of the respective curved worldline. Regarding the curved worldline (2), we can imagine dividing it into an infinite number of infinitesimal linear segments, along each of which the motion of the clock may be considered inertial. Each segment is associated with a local inertial frame relative to which the clock is momentarily at rest.

Since the worldlines (1) and (2) describe real motions at speeds less than c, all elementary spacetime intervals along these lines must be *timelike* [5]:

$$ds^2 = c^2 dt^2 - dx^2 > 0 \quad \Rightarrow \quad ds = (ds^2)^{1/2} \in R \,.$$

Moreover, $ds = c\, d\tau \Rightarrow d\tau = ds/c$, where $d\tau$ is the proper time of the worldline segment, equal to the time measured by a local inertial frame relative to which the corresponding clock is momentarily at rest. So, the *time* interval $d\tau$ measured in the frame of this clock will be equal to the *spacetime* interval ds/c measured in the frame S of the observer relative to whom the two clocks are in motion.

According to either clock the total time between the events A and B is given by the line integral

$$\tau = \int_A^B d\tau = (1/c)\int_A^B ds = (1/c)\int_A^B (c^2 dt^2 - dx^2)^{1/2} \tag{10}$$

(a Lorentz scalar since $d\tau$ is a frame-independent quantity). Of course, the value of the integral (10) depends on the spacetime path from A to B and will be different for the worldlines (1) and (2). As can be proven (see Appendix), among all possible (timelike) worldlines connecting A and B, the unique straight-line path (1) corresponding to *inertial* motion of the associated clock corresponds to the *maximum* proper time τ. The inertial clock (1) will thus measure the *longest* time.

Now, according to the observer using the frame S with coordinates (x, t), the time separation between the events A and B is $\Delta t = t_B - t_A$. Due to the time dilation effect this time interval appears *longer* than the proper time τ measured by the inertial clock (1): $\Delta t > \tau$.

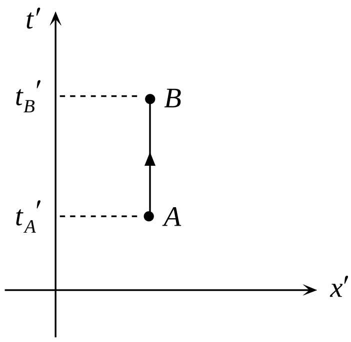

Fig. 2

A particular inertial frame is the frame of the clock (1) itself. Let S' be that frame, with spacetime coordinates (x', t') (see Fig. 2). Since the clock is stationary in S', its worldline in this frame will correspond to x'=*const.* (e.g., x'=0). In this case the proper time τ measured by the clock *is* equal to the time difference $\Delta t' = t_B' - t_A'$. That is, $\Delta t' = \tau$. Of course, $\Delta t' < \Delta t$, where Δt is the time interval in the frame of Fig. 1.

4. The twin paradox

We now come to the "paradox": John and Mary are twins. While John stays at home, his sister goes on a round trip at very high speed (!). When Mary returns home, she finds that John has grown older than her.

Note that John and Mary are *inequivalent* observers, given that John is an *inertial* observer while Mary is *not* (due to the various accelerations she must undergo to perform a round trip). So, John's worldline in an inertial frame will be a straight line, while Mary's worldline will be curved. To simplify matters, we choose this inertial frame to be the rest frame of John himself, with coordinates (x, y, z, t), and we assume that Mary's trip is confined to the x-axis. So, in effect we are dealing with a two-dimensional spacetime with coordinates (x, t). We let (1) and (2) be the worldlines of John and Mary, respectively (see Fig. 3).

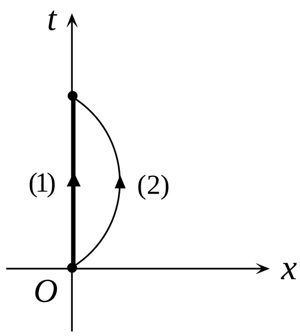

Fig. 3

Let τ_1 and τ_2 be the proper times along the worldlines (1) and (2), equal to the times measured by the clocks of John and Mary, respectively, during Mary's round trip. According to the discussion in Sec. 3, the *longest* proper time corresponds to the clock executing *inertial* motion, which here is John's stationary clock. Therefore $\tau_1 > \tau_2$, which means that John will record a longer time interval. Hence John will be older than Mary when she returns home.

One might now argue that, from Mary's point of view, it is John who has performed a round trip while she has remained stationary. Why then isn't *she* the oldest one at the moment of John's "return"?

The answer is that the preceding analysis was valid only with respect to an *inertial* frame of reference, like that of John's. A similar analysis of the situation from Mary's rest frame would be improper, given that Mary is not an inertial observer. Thus her "conclusion" that she ought to be the older one would simply be wrong! (See [2] for a more thorough discussion.)

Appendix

Consider a line integral of the form

$$Q = \int_{t_1}^{t_2} f\,[x(t),\, x'(t),\, t]\, dt \qquad \text{(A.1)}$$

where f is a given function of the indicated variables (with $x'=dx/dt$) and where $x(t)$ is a curve joining two points (x_1, t_1) and (x_2, t_2) on the plane (x,t); that is,

$$x(t_1) = x_1\,, \qquad x(t_2) = x_2 \qquad \text{(A.2)}$$

We seek the curve $x(t)$ for which the value of the integral Q is an *extremum* (whether this might be a maximum or a minimum). As shown in the calculus of variations (see, e.g., Chap. 2 of [6]) this particular function $x(t)$ satisfies the *Euler-Lagrange equation*

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial x'} = 0 \qquad \text{(A.3)}$$

with the initial condition (A.2).

Let us now go back to the line integral (10), expressing proper time along the worldline of a particle or a clock in two-dimensional spacetime:

$$\tau = (1/c)\int_A^B (c^2 dt^2 - dx^2)^{1/2}$$

where $c^2\, dt^2 - dx^2 > 0$ (timelike interval) and where A and B are fixed spacetime points (events). We have:

$$(c^2 dt^2 - dx^2)^{1/2} = (c^2 - x'^2)^{1/2}\, dt\,.$$

Thus,

$$\tau = (1/c)\int_{t_A}^{t_B} (c^2 - x'^2)^{1/2}\, dt \qquad \text{(A.4)}$$

We seek the spacetime curve $x(t)$ that will make the integral in (A.4) an extremum. To this end, we call

$$(c^2 - x'^2)^{1/2} \equiv f\,(x, x', t)$$

and demand that the differential equation (A.3) be satisfied. Since

$$\frac{\partial f}{\partial x} = 0\,, \qquad \frac{\partial f}{\partial x'} = -\frac{x'}{(c^2 - x'^2)^{1/2}}\,,$$

from (A.3) we have that

$$\frac{x'}{(c^2 - x'^2)^{1/2}} = const. \;\Rightarrow\; x'^2 = const.\,(c^2 - x'^2) \;\Rightarrow\; x'^2 = const.$$

and so $x'(t) = const.$ Thus $x(t)$ is a linear function: $x(t) = \kappa t + \lambda$.

Geometrically, this function describes a *straight* worldline on the plane (x, t) and corresponds to *inertial* motion from A to B. This motion thus corresponds to an *extremum* of the integral in (A.4), thus an extremum of the proper time τ measured by a clock moving from A to B. But, is this extremum a maximum or a minimum?

Let us first note that, physically, x' represents the velocity of a (generally accelerated) clock that moves along a (generally curved) worldline from A to B, in the inertial frame S with spacetime coordinates (x,t): $x' = dx/dt = u$. We thus rewrite (A.4) as

$$\tau = (1/c)\int_{t_A}^{t_B} (c^2 - u^2)^{1/2}\, dt\ .$$

We have: $(c^2 - u^2)^{1/2} = c\,(1 - u^2/c^2)^{1/2} = c\,\gamma(u)^{-1}$, where $\gamma(u) = (1 - u^2/c^2)^{-1/2}$. Hence,

$$\tau = \int_{t_A}^{t_B} \gamma(u)^{-1}\, dt \qquad \text{(A.5)}$$

Let us assume that the frame S is the frame of the inertial observer whose worldline from A to B is a straight line (this choice of frame will not affect the value of τ given that this quantity is a Lorentz scalar). In his own frame S, the observer (and his clock) is stationary, so that u=0, $\gamma(u)$=1 and, by (A.5), $\tau_1 = t_B - t_A$ (see Fig. 4).

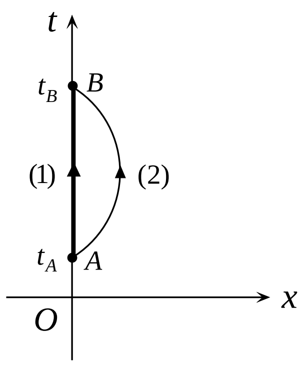

Fig. 4

Any other worldline connecting A with B will necessarily be curved and will represent a clock in accelerated motion. In this case $u \neq 0$ and $\gamma(u)^{-1} < 1$. The integration in (A.5) will thus yield a value $\tau_2 < t_B - t_A$.

Conclusion: The inertial observer's straight worldline corresponds to *maximum* proper time, equal to the actual time measured by the observer's own clock. This observer, therefore, will record the *longest* time between the events A and B.

References

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[1] https://www.researchgate.net/publication/393646118_Aspects_of_Relativity_in_Flat_Spacetime