ON REVISITING THE AXIOMATIC FOUNDATIONS OF NEWTONIAN MECHANICS

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Abstract

In a previous article the axiomatic foundations of Newtonian mechanics were revisited for pedagogical purposes and a certain axiomatic approach to the subject was proposed. Motivated by questions raised by deep-thinking students, this article clarifies some related issues and explains the rationale for seeking a more "economical" axiomatic basis for the theory.

1. Introduction

In a previous article [1] we pointed out certain difficulties in teaching introductory mechanics in a class of deep-thinking students. These difficulties pertain to the very foundations of the theory and manifest themselves in questions like the following:

- Is the law of inertia (Newton's first law) redundant, being no more than a special case of the second law? More generally, do Newton's laws form an independent set of physical principles?
- Is the second law a true law or a definition (of force)?
- Should the third law be considered more fundamental than conservation of momentum, or should it be the other way around?
- Does the "parallelogram rule" for composition of forces on a particle follow trivially from Newton's laws, or is an additional, fourth law required?
- Finally, what is the minimum number of independent laws needed in order to build the entire theory of mechanics?

In Ref. [1] we described an axiomatic and, hopefully, pedagogical approach to fundamental-level mechanics by proposing a "two-dimensional" axiomatic basis for the theory. As independent postulates we chose the *conservation of momentum* and the *independence of interactions* on a particle (leading to the *principle of superposition*). We showed that Newton's laws, as well as all standard ideas of mechanics, follow from these two basic principles.

This theoretical scheme offers plausible answers to students' questions like those listed above. It is our opinion, however, based on various discussions with intelligent students, that more can be said to justify the rationale for revisiting the axiomatic foundations of Newtonian mechanics. This justification is the subject of this article.

2. A critical look at Newton's theory

There have been several attempts to reexamine Newton's laws even since Newton's time. Probably the most important revision of Newton's ideas – and the one on which modern mechanics teaching is based – is that due to Ernst Mach (1838-1916) (for a beautiful discussion of Mach's ideas, see the classic article by H. A. Simon [2]). Our approach differs in several aspects from those of Mach and Simon, although all these approaches share common characteristics in spirit. (For a historical overview of the various viewpoints regarding the theoretical basis of classical mechanics, see, e.g., the first chapter of [3].)

The question of the *independence* of Newton's laws has troubled many generations of physicists. In particular, still on this day some authors assert that the first law (the law of inertia) is but a special case of the second law. The argument goes as follows:

> *"According to the second law, the acceleration of a particle is proportional to the total force acting on it. Now, in the case of a free particle the total force on it is zero. Thus, a free particle must not be accelerating, i.e., its velocity must be constant. But, this is precisely what the law of inertia says!"*

Where is the error in this line of reasoning? Answer: The error rests in regarding the acceleration as an absolute quantity independent of the observer that measures it. As we well know, this is not the case. In particular, the only observer *entitled* to conclude that a non-accelerating object is subject to no net force is an *inertial observer,* one who uses an *inertial frame of reference* for his/her measurements. It is precisely the law of inertia that *defines* inertial frames and *guarantees* their existence. So, without the first law, the second law becomes indeterminate, if not altogether wrong, since it would appear to be valid relative to any observer regardless of his/her state of motion. It may be said that the first law defines the "terrain" within which the second law acquires a meaning. Applying the latter law without taking the former one into account would be like trying to play soccer without possessing a soccer field!

The completeness of Newton's laws is another issue. Let us see a significant example: As is well known, the *principle of conservation of momentum* is a direct consequence of Newton's laws. This principle dictates that the total momentum of a system of particles is constant in time, relative to an inertial frame of reference, when the total external force on the system vanishes (in particular, this is true for an *isolated* system of particles, i.e., a system subject to no external forces). But, when proving this principle we take it for granted that the total force on each particle is the vector sum of all forces (both internal and external) acting on it. This is *not* something that follows trivially from Newton's laws, however! In fact, it was Daniel Bernoulli who first stated this *principle of superposition* after Newton's death. This means that classical Newtonian mechanics is built upon a total of *four* – rather than just three – independent laws.

The question now is: can we somehow "compactify" the axiomatic basis of Newtonian mechanics in order for it to consist of a smaller number of independent principles? At this point it is worth taking a closer look at the principle of conservation of momentum mentioned above. In particular, we note the following:

- For an isolated "system" consisting of a single particle, conservation of momentum reduces to the law of inertia (the momentum, thus also the velocity, of a free particle is constant in time relative to an inertial frame of reference).
- For an isolated system of two particles, conservation of momentum takes us back to the action-reaction law (Newton's third law).

Thus, starting with four fundamental laws (the three laws of Newton plus the principle of superposition) we derived a more general principle (conservation of momentum) that yields, as special cases, two of the laws we started with. The idea is then that, by taking *this* general principle as our fundamental physical law, the number of independent laws necessary for building the theory would be reduced.

How about Newton's second law? We take the view, adopted by several authors including Mach himself (see, e.g., [2,4-7]) that this "law" should be interpreted as simply the *definition* of force as the rate of change of momentum.

We thus end up with a theory built upon *two* fundamental principles, i.e., the conservation of momentum and the principle of superposition. In the following sections these ideas are presented in more detail (see also [1]).

3. The fundamental postulates and their consequences

First some definitions:

- A *frame of reference* (or *reference frame*) is a system of coordinates, or axes, used by an observer to measure physical quantities such as the position, the velocity, the acceleration, etc., of any particle in space. The position of the observer him/herself is assumed *fixed* relative to his/her own reference frame.
- An *isolated system of particles* is a system subject to no *external* interactions, i.e., subject only to their mutual interactions. In particular, an isolated "system" consisting of a single particle is called a *free particle*.

We now state our fundamental postulates, which will henceforth be referred to as *P1* and *P2*:

P1. A class of frames of reference, named *inertial frames*, exists such that, for any *isolated* system of particles, a vector equation of the form

$$\sum_i m_i \vec{v}_i = \text{constant in time} \tag{1}$$

is valid, where $\vec{v}_i$ is the velocity of the particle indexed by i ($i = 1, 2, \cdots$) and where m_i is a constant quantity associated with this particle and independent of the number or nature of interactions the particle is subject to.

We call m_i the *mass* and $\vec{p}_i = m_i \vec{v}_i$ the *momentum* of the ith particle. Also, we call

$$\vec{P} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i \tag{2}$$

the *total momentum* of the system relative to the considered reference frame. Postulate *P1*, then, leads to the *principle of conservation of momentum*:

- *The total momentum of an isolated system of particles, relative to an inertial reference frame, is constant in time.*

This is true, in particular, for a *free* particle.

P2. If a particle of mass m is subject to interactions with particles m_1 , m_2 , ... , then, at each instant t, the rate of change of this particle's momentum relative to an *inertial* reference frame is equal to

$$\frac{d\vec{p}}{dt} = \sum_i \left(\frac{d\vec{p}}{dt} \right)_i \tag{3}$$

where $(d\vec{p}/dt)_i$ is the rate of change of the particle's momentum due solely to the interaction of that particle with the particle m_i (i.e., the rate of change of $\vec{p}$ *if* the particle m interacted *only* with m_i). This postulate expresses the *independence of interactions* in which a particle participates.

A corollary to *P1* states that

- *a free particle moves with constant velocity (i.e., with no acceleration) relative to an inertial reference frame.*

Consequently,

- *any two free particles move with constant velocities relative to each other (their relative velocity is constant and their relative acceleration is zero).*

These statements constitute alternate expressions of the *Law of Inertia* (Newton's first law). Moreover,

- *the position of a free particle may define the origin of an inertial frame of reference.*

An "intelligent" free particle – i.e., one that can make measurements of physical quantities such as velocity or acceleration – is what is called an *inertial observer*. It follows that

- *inertial observers move with constant velocities (i.e., they do not accelerate) relative to one another.*

Consider now a system of two particles of masses m_1 and m_2 . Assume that the particles are allowed to interact with each other for some time interval Δt within which the system may be considered isolated. By conservation of momentum relative to an inertial frame of reference, we have:

$$\Delta(\vec{p}_1+\vec{p}_2)=0 \;\Rightarrow\; \Delta\vec{p}_1=-\Delta\vec{p}_2 \;\Rightarrow\; m_1\,\Delta\vec{v}_1=-m_2\,\Delta\vec{v}_2 \;.$$

We note that the *changes* in the velocities of the two particles within the (arbitrary) time interval Δt are independent of the particular inertial frame used to measure the velocities (although, of course, the velocities themselves *are* frame-dependent!). This is a consequence of the constancy of the relative velocity of any two inertial observers. (The student is invited to explain the above statement analytically.) Taking magnitudes, we have:

$$\frac{|\Delta\vec{v}_1|}{|\Delta\vec{v}_2|}=\frac{m_2}{m_1}=\text{constant} \tag{4}$$

regardless of the kind of interaction or the time Δt. Equation (4) allows us to specify the mass of, say, particle *2* numerically, *relative to* the mass of particle *1* (which particle may arbitrarily be defined to possess a unit mass), by letting the two particles interact for some time. As argued above, the result will be independent of the inertial frame used by the observer who makes the measurements. That is, in the classical theory, *mass is a frame-independent quantity*.

4. The concept of force and the Third Law

We now *define* the *total force* on a particle of mass *m*, at some instant *t*, to be the rate of change of the particle's momentum ($\vec{p}=m\vec{v}$) relative to an *inertial* reference frame, at that instant:

$$\vec{F}=\frac{d\vec{p}}{dt} \tag{5}$$

Since *m* is assumed fixed,

$$\vec{F}=m\vec{a}=m\frac{d\vec{v}}{dt} \tag{6}$$

where $\vec{a}$ is the particle's acceleration at time *t*. Given that both the mass and the acceleration (prove this!) are independent of the inertial frame used to measure them, we conclude that

- *the total force on a particle is a frame-independent quantity.*

With this definition at hand, we can rewrite (3) in the more familiar form

$$\vec{F}=\sum_i \vec{F}_i \tag{7}$$

where $\vec{F}$ is the total force on *m* and $\vec{F}_i$ is the force on *m* due to its interaction with m_i alone. The vector relation (7) expresses the *principle of superposition* referred to in Sec. 2.

We now use *P1* and *P2* to deduce the action-reaction law (Newton's third law). To this end, consider two particles m_1 and m_2. Let $\vec{F}_{12}$ be the force on m_1 due to its interaction with m_2 at time *t*, and let $\vec{F}_{21}$ be the force on m_2 due to its interaction with m_1 at this instant (note the *simultaneity* of action and reaction implicitly assumed here). By the independence of interactions (*P2*) the forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are independent of the presence or not of other particles in interaction with particles m_1 and m_2. Thus, without loss of generality, we may assume that the system of the two particles is isolated. Then, by conservation of momentum and by using (5),

$$\frac{d}{dt}\left(\vec{p}_1+\vec{p}_2\right)=0 \;\Rightarrow\; \frac{d\vec{p}_1}{dt}=-\frac{d\vec{p}_2}{dt} \;\Rightarrow$$

$$\vec{F}_{12}=-\vec{F}_{21} \qquad (8)$$

which is precisely the *Law of Action and Reaction*.

By using (2), (5), (7) and (8), we can prove that (cf. [1])

- *the rate of change of the total momentum of a system of particles at some instant, relative to an inertial reference frame, is equal to the total external force acting on the system at that instant.*

In symbols,

$$\frac{d\vec{P}}{dt}=\sum_i \vec{F}_i=\vec{F}_{ext} \qquad (9)$$

where $\vec{P}$ is the total momentum of the system of particles, defined in equation (2).

5. Discussion of some conceptual problems

After establishing our axiomatic basis and demonstrating that the standard Newtonian laws are consistent with it, the development of the rest of mechanics follows familiar paths. Thus, we can define concepts such as angular momentum, work, kinetic and total mechanical energies, etc., and we can state derivative theorems such as conservation of angular momentum, conservation of mechanical energy, etc. (see [1]). Also, rigid bodies and continuous media can be treated in the usual way [3-9] as systems containing an arbitrarily large number of particles.

Despite the more "economical" axiomatic basis of Newtonian mechanics suggested here, however, certain problems inherent in the classical theory remain. Let us point out a few:

1. The problem of "inertial frames"

An inertial frame of reference is only a theoretical abstraction: such a frame cannot exist in reality. As follows from the discussion in Sec. 3, the origin (say, *O*) of an inertial frame coincides with the position of a hypothetical free particle and, moreover, any real free particle moves with constant velocity relative to *O*. However, no such thing as an absolutely free particle may exist in the world. In the first place, every material particle is subject to the infinitely long-range gravitational interaction with the rest of the world. Furthermore, in order for a supposedly inertial observer to measure the velocity of a "free" particle and verify that this particle is not accelerating relative to him/her, the observer must somehow interact with the particle. Thus, no matter how weak this interaction may be, the particle cannot be considered free in the course of the observation.

2. The problem of simultaneity

In Sec. 4 we used postulates *P1* and *P2*, together with the definition of force, to derive the action-reaction law. Implicit in our arguments was the requirement that action must be simultaneous with reaction. As is well known, this hypothesis, which suggests instantaneous action at a distance, ignores the finite speed of propagation of the field associated with the interaction and violates causality.

3. A dimensionless "observer"

As we have used this concept, an "observer" is an intelligent free particle capable of making measurements of physical quantities such as velocity or acceleration. Such an observer may use any convenient (preferably rectangular) set of axes (x, y, z) for his/her measurements. Different systems of axes used by this observer have different orientations in space. By convention, the observer is located at the origin *O* of the chosen system of axes.

As we know, inertial observers do not accelerate relative to one another. Thus, the relative velocity of the origins (say, *O* and *O′*) of two different inertial frames of reference is constant in time. But, what if the axes of these frames are in *relative rotation* (although the origins *O* and *O′* move uniformly relative to each other, or even coincide)? How can we tell which observer (if any) is an inertial one?

The answer is that, relative to the system of axes of an inertial frame, a free particle does not accelerate. In particular, relative to a rotating frame, a free particle will appear to possess at least a centripetal acceleration. Such a frame, therefore, cannot be inertial.

As mentioned previously, an object with finite dimensions (e.g., a rigid body) can be treated as an arbitrarily large system of particles. No additional postulates are thus needed in order to study the dynamics of such an object. This allows us to regard momentum and its conservation as more fundamental than angular momentum and its conservation, respectively (see also [1]). In this regard, our approach differs significantly from, e.g., that of Simon [2] who, in his own treatment, places the aforementioned two conservation laws on an equal footing from the outset.

References

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[1] See also https://arxiv.org/abs/1205.2326 .

Postulate *P1* $\Longrightarrow$ **Law of Inertia**

Postulate *P2* & Definition of Force $\Longrightarrow$ **Superposition Principle** $\vec{F} = \sum \vec{F}_i$

Postulates *P1 - P2* & Definition of Force $\Longrightarrow$ **Action – Reaction Law** $\vec{F}_{12} = -\vec{F}_{21}$