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Isogroups of Differential Ideals of Vector-Valued Differential Forms: Application to Partial Differential Equations

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Abstract An older geometric technique for the study of invariance groups of partial differential equations, originally proposed by one of the authors and F. B. Estabrook, is generalized and extended to problems involving exterior equations for vector-valued or Lie algebra-valued exterior differential forms. Use of the method is demonstrated in the study of the symmetry groups of the two-dimensional Dirac equation and the full Yang–Mills free-field equations in Minkowski spacetime.

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1. Introduction

This paper presents a certain generalization of an older geometric technique [1] which employs exterior differential forms for the derivation of symmetries of partial differential equations (PDEs). Specifically, the range of applicability of the technique is extended to comprise vector-structured (and, in particular, matrix-structured) PDEs that would otherwise have to be treated by resolution to components. The latter practice often proves to be inconvenient, as the total number of equations constituting the system increases, and so does the number of dependent variables. It is thus desirable in such cases to retain the original (vector) form of the PDEs, which is also a simpler and, typically, more elegant form.

Geometric techniques for symmetry analysis of PDEs were originally proposed by one of the authors (B.K.H.) and F. B. Estabrook [1] in 1971, and later further developed by Edelen [2]. These techniques provide an alternative to other established algebraic formulations (see, for example, the excellent recent book by Olver [3] and the extensive references therein). Both approaches have been successful in treating PDEs for scalar-valued fields. It is our opinion, however, that more can be said about PDEs in which the dependent variables have values in some arbitrary vector space (technically we would say that the equations define sections of some arbitrary vector bundle over the manifold of the independent variables). It turns out that, in such a case, the system of PDEs is geometrically equivalent to a set (actually, a differential ideal) of vector-valued differential forms defined on a manifold with inhomogeneous 'coordinates' (a mixture of scalars and vectors).

There are two technicalities in this more general problem that never emerge in problems for scalar fields. First, when imposing invariance (closure) of the ideal of forms that represent the system, under the action by the Lie derivative, it must be kept in mind that the Lie derivative preserves the degree *and* the specific vector-valuedness of any form on which it operates, and that any form in the ideal must exhibit the same vector-valuedness as the forms that generate the ideal. As it turns out, this forces us to seek automorphisms of the underlying vector space before any actual calculations may begin. Second, if by x^i and y^α we denote the independent and dependent variables, respectively, of the PDEs, and if $F(x^i, y^\alpha)$ is a function of these variables, then the vectorial nature of the y^α generally does *not* permit us to write the exterior derivative of the 0-form F in the usual way as

$$\mathrm{d}F = \frac{\partial F}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial F}{\partial y^\alpha}\,\mathrm{d}y^\alpha \tag{1.1}$$

(the derivatives of F with respect to the y^α are not defined, in general). We thus replace the last term on the right-hand side of Equation (1.1) by a quantity which we denote $\bar{\mathrm{d}}F$ and formally define as the difference between the (total) exterior derivative $\mathrm{d}F$ and the well-defined quantity $\mathrm{d}x^i\,\partial F/\partial x^i$. The operator $\bar{\mathrm{d}}$ can be thought of as an exterior derivative in the fiber space; thus it may appropriately be given the name *internal exterior derivative*. It should be clear that the expression $\bar{\mathrm{d}}F$ may involve the 1-forms $\mathrm{d}y^\alpha$, but not the 1-forms $\mathrm{d}x^i$.

Rather than overwhelm the reader with generalities, we have chosen to illustrate the use of the method through two well-known examples: the two-dimensional Dirac equation (Section 3) which involves multispinor-valued differential forms, and the full classical Yang–Mills free-field equations in Minkowski spacetime (Section 4) which involve Lie algebra-valued forms. Although the calculations in these examples may seem lengthy and involved to the reader, we believe that by no other technique could one derive the symmetries faster and in a less complicated way (see our concluding remarks in Section 5). As noted in Section 5, another example, with new results, is treated in [4].

Two final remarks. First, our attention is focused on the derivation of isovectors of invariance groups, i.e., on symmetry transformations alone; no attempt is made in this article to apply or extend the solution-generating techniques described in [1]. Second, our domain of interest in the present work is restricted to *point* (and, in particular, projectable) transformations. (The question, whether the isogroup of projectable symmetries is exhaustive, is not addressed in this paper.) The search for 'generalized' symmetries by employing geometric techniques is a much more difficult problem, on which research is in progress.

2. Exterior Differential Systems

We are given the following problem: Given a partial differential equation (PDE) or a set of PDEs. Is there an equivalent set of differential forms in involution with respect to the independent variables, i.e., a set whose integral manifolds are solutions of the PDEs?

Cartan [5] has set up criteria for the equivalence of a given set of PDEs (in $n-p$ dependent variables and p independent variables) with a closed set of differential forms on a differentiable n-manifold M. The above set of forms is the basis of a differential ideal of the algebra of forms on M. If an integral manifold N of dimension p exists, we can choose p freely varying variables as coordinates of N and we can functionally specify, in terms of these coordinates, the remaining $(n-p)$ variables. Such an integral manifold then represents geometrically a solution of the original set of PDEs.

Let Γ be a differential ideal of forms defined on M. An *integral manifold* of Γ is a pair (N, ϕ), where N is a submanifold of M and $\phi: N \to M$ is a differentiable map such that the image under the dual map ϕ^*, of any form γ in Γ, vanishes identically: $\phi^* \gamma \equiv 0$. This implies that the forms γ in Γ are annihilated by the tangent vectors of N (actually, by the image of these vectors under the differential map $d\phi \equiv \phi_*$). The integral manifold (N, ϕ) defines a solution of the system of exterior equations $\{\gamma_k = 0\}$, or the system of PDEs $\{\phi^* \gamma_k \equiv 0\}$, where the forms γ_k are a basis of Γ. The forms γ_k need not be of the same degree.

Suppose now we are given a set of *first-order* PDEs in $n-p$ dependent and p independent variables, and let $\{\gamma_k = 0\}$ be the corresponding exterior system. We assume for the moment that the dependent variables in the PDEs are scalars (this includes the possibility that they are components of objects defined on a tensor bundle over the manifold of the independent variables). Let $V(x)$ denote any vector field on the n-dimensional manifold M on which the forms γ_k are defined. Such a field defines a one-parameter group of diffeomorphisms of M [6]. We are interested in a special class of diffeomorphisms, namely those that map integral manifolds of the exterior system into integral manifolds of the same system. This amounts to requiring that the ideal Γ of the forms γ_k be invariant under Lie transport along the integral curves of $V(x)$. This can be arranged by writing

$$\underset{V}{\mathscr{L}} \gamma_i = b_i^k \gamma_k \tag{2.1}$$

where the left-hand side is the Lie derivative of γ_i with respect to V, and where the $b_i^k \equiv b_i^k(x)$ are fields of forms on M. The vector fields $V(x)$ that satisfy Equation (2.1) are called *isovectors* [1, 7] and they are operator realizations of the Lie algebra of the *isogroup* of the ideal Γ (see Appendix for definition of group operators). In classical terms, the isogroup is the group of transformations that leave the original set of PDEs invariant (they preserve their forms and they map solutions into other solutions).

Assume now that the $n-p$ dependent variables can be labelled as $\{y^i_\alpha\}$, where α may denote collectively any set of indices $(\alpha_1\alpha_2\ldots)$ which are distinct from the single index i, where $i=1,\ldots,m$ for some integer m. Assume further that we can find an m-dimensional vector space L with basis $\{e_i\}$ $(i=1,\ldots,m)$, such that (a) the basis vectors e_i are not functions of the coordinates of M and they commute with the elements of any tensor bundle over M, and (b) we can rewrite the original PDEs in terms of a new (and with fewer elements) set of dependent variables defined by

$$y_\alpha = y^i_\alpha e_i. \tag{2.2}$$

Clearly, the y_α are elements of the space L. Now, when the original PDEs are satisfied, the y^i_α are functions of the coordinates $\{x^k\}$ $(k=1,\ldots,p)$ of the submanifold N. Thus the PDEs define sections of a vector bundle over N, the fibers of which are isomorphic to L. On the other hand, if we relax the requirement that the PDEs be satisfied (i.e. if we relax $\gamma_k=0$) then we can define differential forms on M in n (independent) variables. With the introduction of the new variables y_α satisfying conditions (a) and (b), the original set of forms $\{\gamma_k\}$ in n variables can be substituted by a set (with fewer elements) of forms $\{\beta_j\}$ in n' variables $(n'<n)$. These new forms are vector-valued with values in L:

$$\beta_j = \beta^i_j e_i. \tag{2.3}$$

Clearly, $\{\beta^i_j\}=\{\gamma_k\}$. Note also that, by condition (a),

$$\mathrm{d}\beta_j = (\mathrm{d}\beta^i_j)e_i. \tag{2.4}$$

If L has the additional structure of a Lie algebra, then we define

$$\begin{aligned}[\beta_i,\beta_j] &= \beta^k_i \wedge \beta^n_j[e_k, e_n] \\ &= C^m_{kn}\beta^k_i \wedge \beta^n_j e_m \end{aligned} \tag{2.5}$$

and one can show that

$$\mathrm{d}[\beta_i,\beta_j] = [\mathrm{d}\beta_i,\beta_j] + (-1)^q[\beta_i,\mathrm{d}\beta_j] \tag{2.6}$$

where q is the degree of β_i. We note that the space L is in some sense a 'bookkeeping' device which enables us to work with a smaller number n' of variables instead of the original number n. It is not to be confused with the space dual to the space of basis 1-forms underlying the differential forms used here. Contraction of forms with vectors in this dual space, such as Equation (2.10), gives us values in L, hence the term 'vector-valued'.

Introduction of the new variables requires modification of Equation (2.1) to take into account the vector-valuedness of the forms β_k. The generalization of Equation (2.1) will depend on the nature of the problem at hand (see subsequent sections for examples). The general rule is that the action of the Lie derivative on the β_k should not alter their individual tensorial characters. In particular, the right-hand side of the generalization of Equation (2.1) must now accommodate automorphisms of the space $F^{p_i}(M) \otimes L$, where $F^{p_i}(M)$ is the space of all forms on M, of degree p_i equal to the degree of the form β_i on which the Lie derivative operates on the left-hand side. Furthermore, the vector field V is now defined on an n'-dimensional manifold with 'mixed' scalar and L-valued coordinates. Thus we expect that V will have both scalar and L-valued components [8].

As an example, consider the case where all forms γ_k are of the same degree and where the dependent variables in the given PDEs can be expressed as fields with values in gl(N, C) (a problem of this type is the self-dual Yang–Mills equations [4]). Then the β_j will be gl(N, C)-valued, and Equation (2.1) is generalized to

$$\underset{V}{\mathscr{L}} \beta_i = b_i^k \beta_k + A_i^k \beta_k + \beta_k B_i^k \tag{2.7}$$

where the b_i^k are scalars (0-forms) while the A_i^k and B_i^k are in gl(N, C) (0-form matrices). (Note that we have separated the coefficients that commute with the β_j from those that do not commute with the β_j.) The vector V will have both scalar and gl(N, C)-valued components.

In general, the vector V will have a formal representation

$$V = U^i \frac{\partial}{\partial x^i} + W_\alpha \frac{\partial}{\partial y_\alpha} \tag{2.8}$$

where the U^i are scalars and the $W_\alpha = W_\alpha^k e_k$ are L-valued. The literal use of $\partial/\partial y_\alpha$ as a differential operator is limited, due to the L-valuedness of y_α. Quite generally, we define $\partial/\partial y_\alpha$ by the requirement

$$\frac{\partial}{\partial y_\alpha}(A^\beta y_\beta) = A^\alpha \quad \text{when } [y_\alpha, A^\beta] = 0 \quad (\text{all } \beta). \tag{2.9}$$

This definition, although not complete, is sufficient for all applications of interest. In the original variables y_α^k the vector V is written as, by convention,

$$V = U^i \frac{\partial}{\partial x^i} + W_\alpha^k \frac{\partial}{\partial y_\alpha^k} \equiv V^\mu \frac{\partial}{\partial z^\mu} \tag{2.10}$$

where by z^μ ($\mu = 1, \dots, n$) we denote the members of the set of all n variables x^i and y_α^k. The components V^μ will depend on a set of r, say, parameters $a^1, \dots, a^r$. It is possible (at least locally) to arrange a canonical parametrization of V^μ, so that

$$V = a^k V_k^\mu(z) \frac{\partial}{\partial z^\mu} \equiv a^k P_k. \tag{2.11}$$

The quantities $P_k = V_k^\mu \, \partial/\partial z^\mu$ are the infinitesimal operators of the isogroup. They are realizations of the Lie algebra of a group of transformations on the n-dimensional manifold M with coordinates $\{z^\mu\}$. Infinitesimal transformations on M are of the form

$$z^{\mu'} \simeq z^{\mu} - \delta a^k V_k^{\mu}(z) \tag{2.12}$$

(see Appendix; put $V_k^{\mu} = -U_k^{\mu}$ to recover the formulas therein; the sign in front of δa^k is required by group representation theory.) We remark that the relation between Equations (2.11) and (2.12) remains the same in the n' variables $\{x^i, y_\alpha\}$.

3. Dirac Equation

Quite generally, the Dirac equation for a fermion field $\psi(x)$ in the presence of a vector field $A_\mu(x)$ may be written [9]

$$[\gamma^\mu(\partial_\mu - A_\mu) - 1]\psi = 0, \tag{3.1}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$; $\mu = 0, 1, 2, 3$. The x^μ are coordinates in flat spacetime M^4 with the usual signature -2; the A_μ are $n \times n$ complex matrices and the γ^μ are 4×4 matrices. The latter satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} \cdot 1_4 \tag{3.2}$$

where $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Finally, ψ is a $4n$-component object. It can be written as an n-dimensional column vector with entries $\psi_1, \ldots, \psi_n$, where each entry is a four-dimensional complex vector. Thus, at each point x of M^4, $\psi(x)$ has values in a space L which is isomorphic to the tensor product $C^n \otimes C^4$.

To simplify the subsequent algebra, we make the assumption that we live in a two-dimensional submanifold of M^4, say M^2, with coordinates $x^0 \equiv t$, $x^1 \equiv x$, and with signature $g^{\mu\nu} = \mathrm{diag}(1, -1)$. We put $A^\mu \equiv (A^0, A^1) \equiv (\Phi, A)$, so that $A_\mu \equiv (\Phi, -A)$. The Dirac equation then becomes (using a standard notation for derivatives)

$$\gamma^0\psi_t + \gamma^1\psi_x - \gamma^0\Phi\psi + \gamma^1 A\psi - \psi = 0. \tag{3.3}$$

Multiplying by the 2-form $\mathrm{d}t\,\mathrm{d}x$ we obtain the exterior equation

$$\begin{aligned}\eta &\equiv \gamma^0\,\mathrm{d}\psi\,\mathrm{d}x + \gamma^1\,\mathrm{d}t\,\mathrm{d}\psi - (\gamma^0\Phi\psi - \gamma^1 A\psi + \psi)\,\mathrm{d}t\,\mathrm{d}x \\ &\equiv 0 \text{ for solution.}\end{aligned} \tag{3.4}$$

The condition $\eta \equiv 0$ on M^2 is equivalent to the Dirac equation (3.3). We relax this requirement, however, and regard η as a 2-form in five *independent* variables t, x, ψ, Φ, A. (In order to define a differential ideal we must also consider the 3-form $\mathrm{d}\eta$. Since no other 3-forms exist in the system, however, we can disregard $\mathrm{d}\eta$ for the purpose of finding the isovectors.) Clearly, the form η has values in the space L in which ψ takes its values.

To find the isogroup of the ideal, we write

$$\underset{V}{\mathscr{L}}\,\eta = b\eta \tag{3.5}$$

where b is assumed to be a tensor product of an $n \times n$ matrix with a 4×4 matrix

[10] (so that the Lie derivative produces an automorphism of the space of η, as desired). Let V be of the form

$$V = D\frac{\partial}{\partial t} + E\frac{\partial}{\partial x} + B\frac{\partial}{\partial \psi} + F\frac{\partial}{\partial \Phi} + G\frac{\partial}{\partial A}. \tag{3.6}$$

The D and E are scalar functions which we assume to depend only on t and x; B is a $4n$-dimensional column vector, and F and G are $n \times n$ matrices [8].

We now substitute Equation (3.6) and the expression for η into Equation (3.5), using the familiar relations

$$\underset{V}{\mathscr{L}} y^k = V^k; \quad \underset{V}{\mathscr{L}} \mathrm{d}y^k = \mathrm{d}V^k = V^k_{,n}\,\mathrm{d}y^n$$

where $V = V^k\,\partial/\partial y^k$. There is a problem with $\mathscr{L}\,\mathrm{d}\psi = \mathrm{d}B$, since the derivative B_ψ is not defined, in general. We can overcome this difficulty by defining an *internal exterior derivative* (i.e. an exterior derivative that acts only on the fields and not on the spacetime variables) as follows:

$$\bar{\mathrm{d}}f(x^\mu, \psi^i) \equiv \mathrm{d}f - \partial_\mu f\,\mathrm{d}x^\mu \tag{3.7}$$

where f is any function of the spacetime variables x^μ and the fields ψ^i. It is expected that $\bar{\mathrm{d}}f$ will include terms in $\mathrm{d}\psi^i$. Note that, by definition,

$$\bar{\mathrm{d}}f(x^\mu) = 0; \qquad \bar{\mathrm{d}}f(\psi^i) = \mathrm{d}f(\psi^i). \tag{3.8}$$

We can now proceed with Equation (3.5) by putting

$$\underset{V}{\mathscr{L}}\mathrm{d}\psi = \mathrm{d}B = B_t\,\mathrm{d}t + B_x\,\mathrm{d}x + \bar{\mathrm{d}}B,$$

where $\bar{\mathrm{d}}B$ may include terms in $\mathrm{d}\psi$, $\mathrm{d}\Phi$, $\mathrm{d}A$, but not in $\mathrm{d}t$, $\mathrm{d}x$. We thus arrive at an equation involving 2-forms. Equating terms in $\mathrm{d}t\,\mathrm{d}x$ we get

$$\begin{aligned}\gamma^0 B_t &+ \gamma^1 B_x - \gamma^0 F\psi - \gamma^0\Phi B + \gamma^1 G\psi + \gamma^1 AB - B - \\ &- (D_t + E_x)(\gamma^0\Phi\psi - \gamma^1 A\psi + \psi) \\ &= -b(\gamma^0\Phi\psi - \gamma^1 A\psi + \psi).\end{aligned} \tag{3.9}$$

The remaining terms can be separated into those that contain $\mathrm{d}t$ and those that contain $\mathrm{d}x$:

$$(-E_t\gamma^0 + D_t\gamma^1)\,\mathrm{d}t\,\mathrm{d}\psi + \gamma^1\,\mathrm{d}t\,\bar{\mathrm{d}}B = b\gamma^1\,\mathrm{d}t\,\mathrm{d}\psi, \tag{3.10}$$

$$(E_x\gamma^0 - D_x\gamma^1)\,\mathrm{d}\psi\,\mathrm{d}x + \gamma^0\,\bar{\mathrm{d}}B\,\mathrm{d}x = b\gamma^0\,\mathrm{d}\psi\,\mathrm{d}x. \tag{3.11}$$

From the above equations it is clear that $B = B(t, x, \psi)$ and that B must depend linearly on ψ. Thus we put

$$B(t, x, \psi) = e(t, x)\psi + h(t, x), \tag{3.12}$$

where e is a matrix and h is a column vector. Then $\bar{\mathrm{d}}B = e\,\mathrm{d}\psi$, and Equations (3.10)–(3.11) give

$$-E_t\gamma^0 1_n + D_t\gamma^1 1_n + \gamma^1 e = b\gamma^1, \tag{3.13}$$

$$E_x\gamma^0 1_n - D_x\gamma^1 1_n + \gamma^0 e = b\gamma^0, \tag{3.14}$$

while Equation (3.9) becomes an equation of the form $P\psi + Q = R\psi$, which is true for all ψ iff $Q = 0$, $P = R$. Explicitly,

$$\gamma^0 h_t + \gamma^1 h_x - \gamma^0 \Phi h + \gamma^1 A h - h = 0 \tag{3.15}$$

(which says that h is a solution of the Dirac equation (3.3)) and

$$\begin{aligned}&\gamma^0 e_t + \gamma^1 e_x - \gamma^0 F - \gamma^0 \Phi e + \gamma^1 G + \gamma^1 A e - e - \\ &\quad -(D_t + E_x)(\gamma^0\Phi - \gamma^1 A + 1) \\ &= -b(\gamma^0\Phi - \gamma^1 A + 1).\end{aligned} \tag{3.16}$$

Using Equation (3.2) we solve Equation (3.14) for b and substitute the result into Equations (3.13) and (3.16). This gives a pair of partial differential equations, which after pre-multiplication by γ^0 are written

$$E_t 1 + D_t\gamma^0\gamma^1 1_n = E_x\gamma^0\gamma^1 1_n + D_x 1 + [e, \gamma^0\gamma^1], \tag{3.17}$$

$$\begin{aligned}&-e_t + \gamma^0\gamma^1 e_x + 1_4 F + \gamma^0\gamma^1 G + [e, \gamma^0] + \\ &\quad + D_t(1_4\Phi + \gamma^0\gamma^1 A - \gamma^0 1_n) \\ &= D_x(\gamma^0\gamma^1\Phi + 1_4 A - \gamma^1 1_n) + [e, \Phi] + [e, \gamma^0\gamma^1 A].\end{aligned} \tag{3.18}$$

The matrix e is a tensor product of an $n \times n$ matrix with a 4×4 matrix. Thus e can be expanded in a basis of 16 linearly independent 4×4 matrices, the coefficients of expansion being $n \times n$ matrices. We choose as a basis the 16 Dirac matrices Γ^α defined as follows:

$$\Gamma^1 = 1_4,$$

$$\Gamma^{2-5} = \gamma^0, \gamma^1, \gamma^2, \gamma^3 \equiv \gamma^\mu,$$

$$\Gamma^{6-11} = \sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{12}, \sigma^{13}, \sigma^{23} \equiv \sigma^{\mu\nu},$$

$$\Gamma^{12-15} = \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5,$$

$$\Gamma^{16} = \gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma^5,$$

where

$$\sigma^{\mu\nu} = \tfrac{1}{2}[\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu + g^{\mu\nu} \cdot 1_4.$$

We thus make the expansion

$$\begin{aligned}e(t, x) = {}&\alpha 1_4 + \beta^0\gamma^0 + \beta^1\gamma^1 + \beta^2\gamma^2 + \beta^3\gamma^3 + \\ &+ \delta^1\sigma^{01} + \delta^2\sigma^{02} + \delta^3\sigma^{03} + \delta^4\sigma^{12} + \delta^5\sigma^{13} + \delta^6\sigma^{23} + \\ &+ \epsilon^0\gamma^5\gamma^0 + \epsilon^1\gamma^5\gamma^1 + \epsilon^2\gamma^5\gamma^2 + \epsilon^3\gamma^5\gamma^3 + \zeta\gamma^5,\end{aligned} \tag{3.19}$$

where the α, β^μ, δ^k, ϵ^μ, ζ are $n \times n$ matrices which depend on t and x (tensor products between the two types of matrices are assumed in Equation (3.19)). Substituting Equation (3.19) into Equations (3.17) and (3.18), and comparing similar terms in the basis Γ^α, we obtain the following set of equations:

$$\beta^0 = \beta^1 = \beta^2 = \beta^3 = 0, \qquad \delta^2 = \delta^3 = \delta^4 = \delta^5 = 0,$$
$$\epsilon^0 = \epsilon^1 = 0, \qquad \zeta = 0,$$
$$D_t = E_x = 0, \qquad D_x 1_n = E_t 1_n = -2\delta^1,$$
$$q_t - [\Phi, q] = q_x + [A, q] = 0, \quad q \equiv \delta^6, \epsilon^2, \epsilon^3,$$
$$-\alpha_t + F = D_x A + [\alpha, \Phi], \qquad \alpha_x + G = D_x \Phi + [\alpha, A].$$

Remembering that the variables Φ, A are independent of the variables t, x, we integrate the partial differential equations to give

$$D(x) = \omega x + \tau_0, \qquad E(t) = \omega t + \chi_0, \qquad \delta^1 = -\frac{\omega}{2} 1_n,$$
$$q = \text{constant} \cdot 1_n \equiv q \cdot 1_n, \quad q \equiv \delta^6, \epsilon^2, \epsilon^3,$$
$$F(t, x, \Phi, A) = \omega A + [\alpha, \Phi] + \alpha_t,$$
$$G(t, x, \Phi, A) = \omega \Phi + [\alpha, A] - \alpha_x,$$

where the ω, τ_0, χ_0, δ^6, ϵ^2, ϵ^3 are constants, while $\alpha(t, x)$ is a matrix function. The above solutions, together with Equations (3.12), (3.15) and (3.19), give the solution for the vector V of Equations (3.5)–(3.6):

$$\begin{aligned} V = {} & (\omega x + \tau_0)\frac{\partial}{\partial t} + (\omega t + \chi_0)\frac{\partial}{\partial x} + \\ & + \Big[\alpha(t, x)\psi - \frac{\omega}{2}\sigma^{01}\psi + \delta^6\sigma^{23}\psi + \\ & + \epsilon^2\gamma^5\gamma^2\psi + \epsilon^3\gamma^5\gamma^3\psi + h(t, x)\Big]\frac{\partial}{\partial \psi} + \\ & + (\omega A + [\alpha, \Phi] + \alpha_t)\frac{\partial}{\partial \Phi} + \\ & + (\omega\Phi + [\alpha, A] - \alpha_x)\frac{\partial}{\partial A}. \end{aligned} \tag{3.20}$$

Using Equation (2.11), we can write the infinitesimal operators P_k corresponding to the constants ω, τ_0, χ_0, which represent standard Lorentz transformations, time translations, and space translations, respectively. The constants δ^6, ϵ^2 and ϵ^3 represent accidental symmetries of the two-dimensional model, while the presence of the arbitrary solution $h(t, x)$ as an additive factor reflects the linearity of the Dirac equation. Finally, the matrix $\alpha(t, x)$ defines the following infinitesimal transformations:

$$\delta\psi = -\alpha(t, x)\psi, \qquad \delta A_\mu = -[\alpha, A_\mu] - \alpha_{,\mu} \quad (\mu = 0, 1). \tag{3.21}$$

The corresponding finite transformations are

$$\psi' = U\psi, \qquad A'_\mu = UA_\mu U^{-1} - U\,\partial_\mu U^{-1} \tag{3.22}$$

where $U(x^\mu) = \exp\{-\alpha(x^\mu)\}$. These are a general type of gauge transformations. Note that when α is a constant multiple of the identity matrix, then Equation (3.22) is simply a scale change of ψ.

For useful related work on the symmetries of the Dirac equation the reader is referred to [11] and [12]. The latter of these has a very nice treatment of separation of variables for the Dirac equation.

4. Yang–Mills Free-Field Equations

We now study a case where the space L has the structure of a Lie algebra, so that the exterior equations are defined by Lie algebra-valued differential forms. Consider again the Minkowski space M^4 with metric $g_{\mu\nu}$ of signature -2. The Yang–Mills (YM) free-field equations in this space may be written

$$\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = 0, \tag{4.1}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu], \tag{4.2}$$

where the spacetime functions $A^\mu(x)$, $F^{\mu\nu}(x)$ ($x \equiv x^\mu$, $\mu = 0, 1, 2, 3$, in this section) have values in some arbitrary Lie algebra L with basis $\{L_k\}$:

$$A_\mu(x) = A^k_\mu(x) L_k, \qquad F_{\mu\nu}(x) = F^k_{\mu\nu}(x) L_k, \tag{4.3}$$

$$[L_i, L_j] = C^k_{ij} L_k. \tag{4.4}$$

The antisymmetric tensor $F^{\mu\nu}$ has six independent spacetime components. We put

$$(F^{01}, F^{02}, F^{03}, F^{12}, F^{13}, F^{23}) \equiv (F^1, F^2, F^3, F^4, F^5, F^6).$$

Multiplying Equations (4.1) and (4.2) by the 4-form

$$\frac{1}{4!}\epsilon_{\lambda\mu\nu\rho}\,\mathrm{d}x^\lambda\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\mathrm{d}x^\rho = \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

we obtain a set of ten exterior equations which are equivalent to the YM system. Stated differently, we define a set of ten 4-forms (in 14 variables) whose restriction to M^4 is required to vanish identically in order that the YM equations be satisfied:

$$\gamma_1 = \mathrm{d}t\,\mathrm{d}F^1\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}F^2\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}F^3 + \\ +([A^1, F^1] + [A^2, F^2] + [A^3, F^3])\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\gamma_2 = \mathrm{d}F^1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}F^4\,\mathrm{d}z - \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}F^5 - \\ - ([A^0, F^1] + [A^2, F^4] + [A^3, F^5])\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\gamma_3 = \mathrm{d}F^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}F^4\,\mathrm{d}y\,\mathrm{d}z - \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}F^6 + \\ + (-[A^0, F^2] + [A^1, F^4] - [A^3, F^6])\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\gamma_4 = \mathrm{d}F^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}F^5\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}F^6\,\mathrm{d}z + \\ + (-[A^0, F^3] + [A^1, F^5] + [A^2, F^6])\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\gamma_5 = \mathrm{d}A^1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}A^0\,\mathrm{d}y\,\mathrm{d}z - (F^1 + [A^0, A^1])\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\gamma_6 = \mathrm{d}A^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}A^0\,\mathrm{d}z - (F^2 + [A^0, A^2])\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\gamma_7 = \mathrm{d}A^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}A^0 - (F^3 + [A^0, A^3])\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\gamma_8 = -\mathrm{d}t\,\mathrm{d}A^2\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}A^1\,\mathrm{d}z - (F^4 + [A^1, A^2])\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\gamma_9 = -\mathrm{d}t\,\mathrm{d}A^3\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}A^1 - (F^5 + [A^1, A^3])\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\gamma_{10} = -\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}A^3\,\mathrm{d}z + \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}A^2 - (F^6 + [A^2, A^3])\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

It can be shown that the forms $\mathrm{d}\gamma_k$ are in the ideal of the γ_k (note, for example, that $\mathrm{d}\gamma_5 = \mathrm{d}t(\gamma_2 + [A^0, \gamma_5]) + \mathrm{d}x[\gamma_5, A^1]$). Thus this ideal is closed.

The action of the Lie derivative on the γ_k must preserve the Lie algebra-valuedness of these forms. We are thus seeking automorphisms of L. For this purpose we write

$$\underset{V}{\mathcal{L}}\gamma_i = a_i^k\gamma_k + \mathrm{Ad}(b_i^k)\gamma_k \\ = a_i^k\gamma_k + [b_i^k, \gamma_k], \tag{4.5}$$

where the a_i^k are scalar functions, the b_i^k are L-valued functions, and where $\mathrm{Ad}(b)$ denotes an operator associated with the adjoint representation of L for some element b of L. We seek symmetries for which the a_i^k and the b_i^k may depend on the x^μ, but not on the A^μ and F^k (this means that the parameters of the symmetry group are, at most, functions of x).

We write V as

$$V = D^\mu(x)\frac{\partial}{\partial x^\mu} + B^\mu(x, A^\nu, F^k)\frac{\partial}{\partial A^\mu} + \\ + G^i(x, A^\nu, F^k)\frac{\partial}{\partial F^i}, \tag{4.6}$$

where the D^μ are scalars, while the B^μ and G^i are L-valued. Substituting Equation (4.6) into Equation (4.5), and using Equation (3.7) to write

$$\mathrm{d}B^\mu = B^\mu_{,\lambda}\,\mathrm{d}x^\lambda + \bar{\mathrm{d}}B^\mu, \qquad \mathrm{d}G^k = G^k_{,\lambda}\,\mathrm{d}x^\lambda + \bar{\mathrm{d}}G^k,$$

we obtain a set of 10 equations involving 4-forms. We proceed by equating the coefficients of $dt\,dx\,dy\,dz$ on both sides of each equation. This again gives a set of 10 equations which we write below. For brevity we do not display the right-hand sides explicitly, but we introduce the symbol Γ_k to denote the coefficient of $dt\,dx\,dy\,dz$ in γ_k:

$$\begin{aligned}
&G^1_x + G^2_y + G^3_z + D^{\mu}_{,\mu}([A^1, F^1] + [A^2, F^2] + [A^3, F^3]) + \\
&\quad +[B^1, F^1] + [A^1, G^1] + [B^2, F^2] + [A^2, G^2] + [B^3, F^3] + [A^3, G^3] \\
&\quad = a^k_1\Gamma_k + [b^k_1, \Gamma_k], \\
&G^1_t - G^4_y - G^5_z - D^{\mu}_{,\mu}([A^0, F^1] + [A^2, F^4] + [A^3, F^5]) - \\
&\quad -([B^0, F^1] + [A^0, G^1] + [B^2, F^4] + [A^2, G^4] + [B^3, F^5] + [A^3, G^5]) \\
&\quad = a^k_2\Gamma_k + [b^k_2, \Gamma_k], \\
&G^2_t + G^4_x - G^6_z + D^{\mu}_{,\mu}(-[A^0, F^2] + [A^1, F^4] - [A^3, F^6]) - \\
&\quad -[B^0, F^2] - [A^0, G^2] + [B^1, F^4] + [A^1, G^4] - [B^3, F^6] - [A^3, G^6] \\
&\quad = a^k_3\Gamma_k + [b^k_3, \Gamma_k], \\
&G^3_t + G^5_x + G^6_y + D^{\mu}_{,\mu}(-[A^0, F^3] + [A^1, F^5] + [A^2, F^6]) - \\
&\quad -[B^0, F^3] - [A^0, G^3] + [B^1, F^5] + [A^1, G^5] + [B^2, F^6] + [A^2, G^6] \\
&\quad = a^k_4\Gamma_k + [b^k_4, \Gamma_k], \\
&B^1_t + B^0_x - D^{\mu}_{,\mu}(F^1 + [A^0, A^1]) - (G^1 + [B^0, A^1] + [A^0, B^1]) \\
&\quad = a^k_5\Gamma_k + [b^k_5, \Gamma_k], \\
&B^2_t + B^0_y - D^{\mu}_{,\mu}(F^2 + [A^0, A^2]) - (G^2 + [B^0, A^2] + [A^0, B^2]) \\
&\quad = a^k_6\Gamma_k + [b^k_6, \Gamma_k], \\
&B^3_t + B^0_z - D^{\mu}_{,\mu}(F^3 + [A^0, A^3]) - (G^3 + [B^0, A^3] + [A^0, B^3]) \\
&\quad = a^k_7\Gamma_k + [b^k_7, \Gamma_k], \\
&-B^2_x + B^1_y - D^{\mu}_{,\mu}(F^4 + [A^1, A^2]) - (G^4 + [B^1, A^2] + [A^1, B^2]) \\
&\quad = a^k_8\Gamma_k + [b^k_8, \Gamma_k], \\
&-B^3_x + B^1_z - D^{\mu}_{,\mu}(F^5 + [A^1, A^3]) - (G^5 + [B^1, A^3] + [A^1, B^3]) \\
&\quad = a^k_9\Gamma_k + [b^k_9, \Gamma_k], \\
&-B^3_y + B^2_z - D^{\mu}_{,\mu}(F^6 + [A^2, A^3]) - (G^6 + [B^2, A^3] + [A^2, B^3]) \\
&\quad = a^k_{10}\Gamma_k + [b^k_{10}, \Gamma_k].
\end{aligned} \tag{4.7}$$

We now put

$$\begin{aligned}
B^{\mu} &= \alpha^{\mu\nu}(x)A^{\nu} + \beta^{\mu k}(x)F^k + \bar{B}^{\mu}(x, A^{\nu}, F^k), \\
G^i &= \delta^{i\mu}(x)A^{\mu} + \epsilon^{ik}(x)F^k + \bar{G}^i(x, A^{\nu}, F^k),
\end{aligned} \tag{4.8}$$

where the α, β, δ, ϵ are scalars (note the summations). Then

$$\bar{\mathrm{d}} B^{\mu} = \alpha^{\mu\nu}\, \mathrm{d}A^{\nu} + \beta^{\mu k}\, \mathrm{d}F^{k} + \bar{\mathrm{d}}\bar{B}^{\mu},$$

$$\bar{\mathrm{d}} G^{i} = \delta^{i\mu}\, \mathrm{d}A^{\mu} + \epsilon^{ik}\, \mathrm{d}F^{k} + \bar{\mathrm{d}}\bar{G}^{i}.$$

After these expansions are made, the remaining terms in the 10 equations obtained by expanding Equation (4.5) can be divided into (a) those that are *scalar* multiples of the various basis 4-forms (i.e., multiples in which the coefficients commute with the basis 4-forms), and (b) those (terms) that are not of the type (a).

There are 40 different terms of type (a) (we do not include terms in $\mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ that were taken care of earlier). Comparison of coefficients of similar terms thus yields a set of 400 equations (certain of which are trivial identities). Using these equations we can eliminate the a_i^k from our problem by expressing them in terms of other quantities. By this process we also obtain a number of algebraic and differential equations for unknown quantities. (We regret that we cannot display all of our results explicitly, but this would be practically impossible for a paper of average length. We invite the interested reader to undertake the filling-in of missing steps as an instructive exercise.) From the 400 equations mentioned above we also obtain information about the coefficients α, β, δ, ϵ in Equation (4.8):

$$\beta^{\mu k} = 0, \quad \text{all } \mu, k;$$

$$\delta^{12} = -\delta^{21} = -\delta^{40} \equiv \delta^{1}, \qquad \delta^{13} = -\delta^{31} = -\delta^{50} \equiv \delta^{2},$$

$$\delta^{23} = -\delta^{32} = -\delta^{60} \equiv \delta^{3}, \qquad \delta^{43} = -\delta^{52} = \delta^{61} \equiv \delta^{4};$$

all other $\delta^{i\mu}$ are zero;

$$\alpha^{\mu\nu} = kD^{\nu}_{,\mu}, \quad \mu \neq \nu,$$

where $k = +1$ or -1 according as the product $\mu\nu = 0$ or $\neq 0$ (the $\alpha^{\mu\nu}$ with $\mu = \nu$ will be specified later);

$$\epsilon^{16} = \epsilon^{61} = \epsilon^{25} = \epsilon^{52} = \epsilon^{34} = \epsilon^{43} = 0,$$

$$\epsilon^{12} = \epsilon^{56} = D^{1}_{y}, \qquad \epsilon^{13} = -\epsilon^{46} = D^{1}_{z},$$

$$\epsilon^{14} = -\epsilon^{36} = -D^{0}_{y}, \qquad \epsilon^{15} = \epsilon^{26} = -D^{0}_{z},$$

$$\epsilon^{21} = \epsilon^{65} = D^{2}_{x}, \qquad \epsilon^{23} = \epsilon^{45} = D^{2}_{z},$$

$$\epsilon^{24} = \epsilon^{35} = D^{0}_{x}, \qquad \epsilon^{31} = -\epsilon^{64} = D^{3}_{x},$$

$$\epsilon^{32} = \epsilon^{54} = D^{3}_{y}, \qquad \epsilon^{41} = -\epsilon^{63} = -D^{2}_{t},$$

$$\epsilon^{42} = \epsilon^{53} = D^{1}_{t}, \qquad \epsilon^{51} = \epsilon^{62} = -D^{3}_{t}$$

(the ϵ^{ik} with $i = k$ will be specified later).

For the coefficients a_i^k we have:

$$a_1^1 = D_t^0 + D_x^1 + D_y^2 + \epsilon^{33} = D_t^0 + D_x^1 + D_z^3 + \epsilon^{22} = D_t^0 + D_y^2 + D_z^3 + \epsilon^{11},$$
$$a_2^2 = D_t^0 + D_x^1 + D_y^2 + \epsilon^{55} = D_t^0 + D_x^1 + D_z^3 + \epsilon^{44} = D_x^1 + D_y^2 + D_z^3 + \epsilon^{11},$$
$$a_3^3 = D_t^0 + D_x^1 + D_y^2 + \epsilon^{66} = D_t^0 + D_y^2 + D_z^3 + \epsilon^{44} = D_x^1 + D_y^2 + D_z^3 + \epsilon^{22},$$
$$a_4^4 = D_t^0 + D_x^1 + D_z^3 + \epsilon^{66} = D_t^0 + D_y^2 + D_z^3 + \epsilon^{55} = D_x^1 + D_y^2 + D_z^3 + \epsilon^{33},$$
$$a_5^5 = D_t^0 + D_y^2 + D_z^3 + \alpha^{00} = D_x^1 + D_y^2 + D_z^3 + \alpha^{11},$$
$$a_6^6 = D_t^0 + D_x^1 + D_z^3 + \alpha^{00} = D_x^1 + D_y^2 + D_z^3 + \alpha^{22},$$
$$a_7^7 = D_t^0 + D_x^1 + D_y^2 + \alpha^{00} = D_x^1 + D_y^2 + D_z^3 + \alpha^{33},$$
$$a_8^8 = D_t^0 + D_x^1 + D_z^3 + \alpha^{11} = D_t^0 + D_y^2 + D_z^3 + \alpha^{22},$$
$$a_9^9 = D_t^0 + D_x^1 + D_y^2 + \alpha^{11} = D_t^0 + D_y^2 + D_z^3 + \alpha^{33},$$
$$a_{10}^{10} = D_t^0 + D_x^1 + D_y^2 + \alpha^{22} = D_t^0 + D_x^1 + D_z^3 + \alpha^{33},$$

$$-a_1^2 = a_8^6 = a_9^7 = D_x^0, \qquad -a_1^3 = -a_8^5 = a_{10}^7 = D_y^0,$$
$$a_1^4 = a_9^5 = a_{10}^6 = -D_z^0, \qquad -a_2^1 = a_6^8 = a_7^9 = D_t^1,$$
$$a_2^3 = -a_6^5 = -a_{10}^9 = D_y^1, \qquad a_2^4 = -a_7^5 = a_{10}^8 = D_z^1,$$
$$-a_3^1 = -a_5^8 = a_7^{10} = D_t^2, \qquad a_3^2 = -a_5^6 = -a_9^{10} = D_x^2,$$
$$a_3^4 = -a_7^6 = -a_9^8 = D_z^2, \qquad a_4^1 = a_5^9 = a_6^{10} = -D_t^3,$$
$$a_4^2 = -a_5^7 = a_8^{10} = D_x^3, \qquad a_4^3 = -a_6^7 = -a_8^9 = D_y^3,$$
$$-a_1^8 = a_2^6 = -a_3^5 = \delta^1, \qquad -a_1^9 = a_2^7 = -a_4^5 = \delta^2,$$
$$-a_1^{10} = a_3^7 = -a_4^6 = \delta^3, \qquad a_2^{10} = -a_3^9 = a_4^8 = \delta^4;$$

the remaining a_i^k are zero.

We now turn to the terms of type (b) in the expansion of Equation (4.5). These terms can be divided into four kinds, according to their dependence on the basis 3-forms $\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$, $\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}z$, $\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z$ or $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. The L-valued coefficients of each of these basis 3-forms must be equated in each of the 10 exterior equations, the process thus yielding a set of 40 equations for 1-forms. These equations are of two general types:

$$[C_k, \mathrm{d}Y^k] = 0 \tag{4.9}$$

and

$$\bar{\mathrm{d}}\bar{H}^i = [e_k^i, \mathrm{d}Y^k], \tag{4.10}$$

where the C_k and e_k^i are elements of the set $\{\pm b_i^j\}$ (note however that the indexing of the e^i_k is totally irrelevant to that of the b^j_i), where the Y^k denote the A^μ and F^k, collectively, and where the H^i denote the B^μ and G^k, collectively. Given the independence of the 1-forms $\mathrm{d}Y^k$, and the fact that the C_k and Y^k do not commute, Equation (4.9) implies $C_k = 0$, all k. Also, given that, by definition, $\mathrm{d}Y^k = \bar{\mathrm{d}}Y^k$, $\bar{\mathrm{d}}e^i_k(x) = 0$, Equation (4.10) can be integrated immediately:

$$\bar{H}^i = [e^i_k, Y^k] + h^i(x),$$

where $h^i(x)$ is an arbitrary function. We now state our results explicitly:

$$b^1_1 = b^2_2 = b^3_3 = b^4_4 \equiv b^1, \qquad b^5_5 = b^6_6 = b^7_7 = b^8_8 = b^9_9 = b^{10}_{10} \equiv b^2,$$

$$b^8_1 = -b^6_2 = b^5_3 \equiv b^3, \qquad b^9_1 = -b^7_2 = b^5_4 \equiv b^4,$$

$$b^{10}_1 = -b^7_3 = b^6_4 \equiv b^5, \qquad b^{10}_2 = -b^9_3 = b^8_4 \equiv b^6;$$

all other b^k_i are zero;

$$\bar{B}^\mu = [b^2, A^\mu] + \lambda^\mu(x), \qquad \bar{G}^k = [b^1, F^k] + g^k(x) + J^k,$$

where the $\lambda^\mu(x)$ and $g^k(x)$ are arbitrary functions, and where

$$J^1 = -[b^3, A^2] - [b^4, A^3], \qquad J^2 = [b^3, A^1] - [b^5, A^3],$$

$$J^3 = [b^4, A^1] + [b^5, A^2], \qquad J^4 = [b^3, A^0] + [b^6, A^3],$$

$$J^5 = [b^4, A^0] - [b^6, A^2], \qquad J^6 = [b^5, A^0] + [b^6, A^1].$$

Making the appropriate substitutions into Equation (4.8), we obtain expressions for B^μ and G^i which we substitute back into the 10 equations (4.7). The coefficients a^k_i appearing in these equations can be eliminated in favor of other quantities, as we have seen already, while certain substitutions can also be made with regard to the b^k_i. The result is a set of equalities between expressions that can be loosely described as a generalized type of 'polynomials' in the variables A^μ and F^k, with x-dependent coefficients. The 'constant' term in such a 'polynomial' is a matrix function $f(x)$, while the other terms are of the following kinds: qA^μ, qF^k, $q[A^\mu, A^\nu]$, $q[A^\mu, F^k]$, $[Q, A^\mu]$, $[Q, F^k]$, $[Q, [A^\mu, A^\nu]]$, $[Q, [A^\mu, F^k]]$, where $q(x)$ is a scalar function while $Q(x)$ is an L-valued function. Coefficients of similar terms are then equated in each of the 10 polynomial equations, this process yielding an enormous set of algebraic and partial differential equations which, however, are not hard to solve. Indeed, by a straightforward procedure, we find:

$$\delta^1 = \delta^2 = \delta^3 = \delta^4 = 0;$$

$$b^1 = b^2 \equiv b(x), \qquad b^3 = b^4 = b^5 = b^6 = 0;$$

$$g^k(x) = 0, \quad k = 1, \ldots, 6;$$

$$\lambda^\mu(x) = \partial^\mu b(x) = g^{\mu\nu} b_{,\nu}.$$

We also obtain the following set of partial differential equations:

$$D^0_t = D^1_x = D^2_y = D^3_z \equiv \omega^0,$$

$$D^0_x = D^1_t \equiv \omega^1, \qquad D^0_y = D^2_t \equiv \omega^2, \qquad D^0_z = D^3_t \equiv \omega^3,$$

$$D^1_y = -D^2_x \equiv \omega^4, \qquad D^1_z = -D^3_x \equiv \omega^5, \qquad D^2_z = -D^3_y \equiv \omega^6,$$

where the ω^i are functions of x. Finally, we have the relations

$$\alpha^{00} = \alpha^{11} = \alpha^{22} = \alpha^{33} = -\omega^0,$$

$$\epsilon^{11} = \epsilon^{22} = \epsilon^{33} = \epsilon^{44} = \epsilon^{55} = \epsilon^{66} = -2\omega^0.$$

The general solutions of the partial differential equations can be found by a power series expansion of the D^μ. The solutions are

$$\begin{aligned}
D^0 &= c_0 + \alpha t + \alpha_1 x + \alpha_2 y + \alpha_3 z + \\
&\quad + \beta_0(t^2 + x^2 + y^2 + z^2) - 2(\beta_1 tx + \beta_2 ty + \beta_3 tz),\\
D^1 &= c_1 + \alpha x + \alpha_1 t + \alpha_4 y + \alpha_5 z - \\
&\quad - \beta_1(t^2 + x^2 - y^2 - z^2) + 2(\beta_0 tx - \beta_2 xy - \beta_3 xz),\\
D^2 &= c_2 + \alpha y + \alpha_2 t - \alpha_4 x + \alpha_6 z - \\
&\quad - \beta_2(t^2 - x^2 + y^2 - z^2) + 2(\beta_0 ty - \beta_1 xy - \beta_3 yz),\\
D^3 &= c_3 + \alpha z + \alpha_3 t - \alpha_5 x - \alpha_6 y - \\
&\quad - \beta_3(t^2 - x^2 - y^2 + z^2) + 2(\beta_0 tz - \beta_1 xz - \beta_2 yz),
\end{aligned}$$

where the c_μ, α, α_k, β_μ are real constants. Having the D^μ at our disposal, we can now evaluate the quantities ω^k ($k = 0, 1, \ldots, 6$) and use them, in turn, to evaluate the $\alpha^{\mu\nu}$ and ϵ^{ik} of Equation (4.8) (recall that the $\beta^{\mu k}$ and $\delta^{i\mu}$ are zero). Then, given that we have already obtained expressions for the B^μ and G^i, we can write the solutions for B^μ and G^i. We leave this straightforward construction to the reader, and we urge her or him to check that the result includes the correct transformations of the tensors A^μ and $F^{\mu\nu}$, corresponding to the infinitesimal coordinate changes [13]

$$x^{\mu'} \simeq x^\mu - D^\mu(x; \delta c_\mu, \delta\alpha, \delta\alpha_k, \delta\beta_\mu)$$

where the parameters δc_μ, etc., are infinitesimal versions of the previously defined parameters. We remark that the general forms of the solutions for B^μ and $G^i \equiv G^{\mu\nu}$ ($\mu < \nu$) found in this way are

$$B^\mu = Q^\mu(A^\nu; \alpha, \alpha_k, \beta_\lambda) + [b(x), A^\mu] + \partial^\mu b(x),$$

$$G^{\mu\nu} = R^{\mu\nu}(F^{\lambda\rho}; \alpha, \alpha_k, \beta_\lambda) + [b(x), F^{\mu\nu}],$$

where the Q^μ and $R^{\mu\nu}$ are linear functions of the A^ν and $F^{\lambda\rho}$, respectively, and where $Q^\mu = R^{\mu\nu} = 0$ for $\alpha = \alpha_k = \beta_\lambda = 0$. The case $Q^\mu = R^{\mu\nu} = 0$ corresponds to the gauge transformation (cf. Equations (3.21) and (3.22))

$$A'_\mu = UA_\mu U^{-1} - U\partial_\mu U^{-1}, \qquad F'_{\mu\nu} = UF_{\mu\nu}U^{-1} \tag{4.11}$$

where $U(x) = \exp\{-b(x)\}$ and $b(x) = -\theta^k(x)L_k$, the θ^k being real, x-dependent parameters, and the L_k being the basis of the Lie algebra L.

We now find expressions for the infinitesimal group operators. The vector V of Equations. (4.5) and (4.6) is parametrized by the 15 real parameters c_μ, α, $\alpha_k \equiv \alpha_{\mu\nu}$ $(\mu < \nu)$, β_μ, and by the real parameters $\theta^k(x)$. The 15-parameter subgroup corresponds to coordinate transformations, so it is sufficient to express the corresponding 15 operators in the basis $\{\partial/\partial x^\mu\}$. From the solutions for D^μ and from Equation (2.11), we find, in the notation of Equation (A4) of the Appendix:

$$P_{c_\mu} = \frac{\partial}{\partial x^\mu} \equiv \partial_\mu, \qquad P_\alpha = x^\mu \partial_\mu,$$

$$P_{\alpha_{\mu\nu}} = x_\mu \partial_\nu - x_\nu \partial_\mu \quad (\mu < \nu), \qquad P_{\beta_\mu} = 2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu .$$

These are readily identified as the infinitesimal operators of the conformal group. Conformal invariance is a familiar property of the Yang–Mills equations.

Let V_g be the part of the vector V that corresponds to the internal (gauge) transformations of Equation (4.11). We write, in the spirit of Equations (2.8) and (2.10),

$$\begin{aligned} V_g &= ([b, A_\mu] + \partial_\mu b)\frac{\partial}{\partial A_\mu} + [b, F_{\mu\nu}]\frac{\partial}{\partial F_{\mu\nu}} \\ &= -(C^k_{ij}\theta^i A^j_\mu + \theta^k_{,\mu})\frac{\partial}{\partial A^k_\mu} - C^k_{ij}\theta^i F^j_{\mu\nu}\frac{\partial}{\partial F^k_{\mu\nu}}, \end{aligned}$$

where we have used Equations (4.3) and (4.4). From the above expression we can read off the operators corresponding to the parameters θ^i and $\theta^k_{,\mu}$:

$$P_i = -C^k_{ij}\left(A^j_\mu \frac{\partial}{\partial A^k_\mu} + F^j_{\mu\nu}\frac{\partial}{\partial F^k_{\mu\nu}}\right),$$

$$P^\mu_k = -\frac{\partial}{\partial A^k_\mu}.$$

With the aid of the Jacobi identity, the reader may show that the P_i satisfy isomorphic commutation relations with the L_i (Equation (4.4)). We thus conclude that the P_i constitute a realization of the Lie algebra L. A comparison with Equation (A6′) shows that these operators are associated with the action of a Lie group, via its adjoint representation, on a manifold with coordinates A^k_μ and $F^k_{\mu\nu}$.

5. Concluding Remarks

As a conclusion to this paper we would like to make a few comments on the usefulness of the method described in the previous sections. The advantages of this method (as we perceive them) are manifest on both the conceptual and the practical or computational levels. At the conceptual level, new insights are gained on the transformation character of certain geometrical objects with complex tensorial structures. At the computational level, the technique provides a faster and more compact way to derive symmetries.

The reader may question the validity of the last statement above, in view, for example, of the length of the previous section on the Yang–Mills equations. Let us thus briefly compare our present approach to the aforementioned problem with other approaches at our disposal.

First, let us sketch one possible treatment based on the original technique as proposed in [1]. To begin with, Equations (4.1) and (4.2) must be written in component form by using Equations (4.3) and (4.4):

$$\partial_\mu F_k^{\mu\nu} + C_{ik}^j A_\mu^i F_j^{\mu\nu} = 0, \tag{5.1}$$

$$F_{\mu\nu}^k = \partial_\mu A_\nu^k - \partial_\nu A_\mu^k - C_{ij}^k A_\mu^i A_\nu^j, \tag{5.2}$$

where the i, j, k run from 1 to n, the latter denoting the dimension of the Lie algebra L. Thus, we have a total of $10n$ equations, which may be represented by a set of $10n$ 4-forms ω_k in the $(4+10n)$ variables x^μ, A_k^μ and $F_k^{\mu\nu}$. We then demand that

$$\underset{V}{\mathscr{L}} \omega_i = a_i^k \omega_k, \tag{5.3}$$

where the a_i^k are scalar quantities and where

$$V = D^\mu \frac{\partial}{\partial x^\mu} + B_k^\mu \frac{\partial}{\partial A_k^\mu} + G_k^i \frac{\partial}{\partial F_k^i} \tag{5.4}$$

(here we have re-labelled the $F_k^{\mu\nu}$ as F_k^i).

Substituting Equation (5.4) into Equation (5.3) we obtain a system of $10n$ exterior equations, which involve a total of $1+40n$ different basis 4-forms. Equating coefficients of similar terms we finally obtain a system of $10n+400n^2$ differential and algebraic equations (some of which will be trivial identities). Using these relations we can then eliminate the coefficients a_i^k and solve for the components of the vector V.

Given that for the lowest-order non-Abelian gauge group, SU(2), it is $n=3$, we see that such an approach would require the solution of a minimum of 3630 equations! (In fact, there are now computer programs that do just this.) We now see the advantage of using the generalized method: we were able to calculate the symmetries of the YM system for *any* gauge group, no matter how large its order. ('Accidental' or 'hidden' symmetries, associated with a particular gauge group, can be found only by allowing generalized vector fields, in the sense of [3].)

As a second alternative, the reader may try using the algebraic techniques described in Chapter 2 of [3] (remember that we are concerned with point transformations only). In this approach we again write the YM equations in their component form (5.1)–(5.2), but now represent this system as a set of algebraic relations

$$\Delta_k^\nu(A_i^\mu, \partial_\lambda A_i^\mu, \partial_\lambda\partial_\rho A_i^\mu) = 0$$

in the variables indicated, where $\nu = 0, 1, 2, 3$ and $k = 1, \ldots, n$, and where the quantities Δ_k^ν can be easily inferred from Equations (5.1)–(5.2). We then define the vector field

$$V = D^\mu \frac{\partial}{\partial x^\mu} + B_k^\mu \frac{\partial}{\partial A_k^\mu},$$

construct its second prolongation $\mathrm{pr}^{(2)} V$ (which is by no means an easy task), and demand that

$$\mathrm{pr}^{(2)} V(\Delta_k^\nu) = 0 \quad \text{when } \Delta_k^\nu = 0.$$

It turns out then that the coefficients of a large number of monomials in A_i^μ and its derivatives must vanish, which eventually gives an enormous set of partial differential equations for the coefficients of V. Again, the higher the order of the gauge group, the more cumbersome the problem becomes.

One final word. It seems to us that there are problems in which it is absolutely important to retain the vector or matrix structure of the given PDEs. An example is the self-dual Yang–Mills equation in the J formulation, discussed in [4]. (The reader is invited to try this problem in component form, using any method she or he prefers.) It is remarkable that, in this problem, the generalized isovector approach gives, besides symmetry transformations, an explicit construction of a linear system, as well as of infinitesimal Bäcklund transformations for the self-dual system.

6. Appendix: Group Operators

Consider first an n-dimensional Lie group G of transformations on an m-dimensional manifold M. Denote by $\{x^1, \ldots, x^m\}$ a local set of coordinates on M and by $\{a^1, \ldots, a^n\}$ a real, faithful parametrization of G near the identity (the identity, by convention, corresponds to $a^k = 0$, all k.) For every g in G define the operator $T(g)$, acting on functions F on M, by

$$[T(g)F](\mathbf{x}) \equiv F(g^{-1}\mathbf{x}) \tag{A1}$$

where $\mathbf{x}$ is a point on M, and where $g\mathbf{x}$ generally denotes the action of G on M. Infinitesimally the group transformations can be expressed as

$$x^{\mu'} \simeq x^\mu + \delta a^k U_k^\mu(\mathbf{x}), \tag{A2}$$

with $\mu = 1, \ldots, m$; $k = 1, \ldots, n$. Therefore

$$\begin{aligned}[T(g)F](\mathbf{x}) &\simeq F(x^\mu - \delta a^k U_k^\mu) \\ &\simeq [(1 - \delta a^k U_k^\mu \partial_\mu)F](\mathbf{x}) \\ &\equiv [(1+V)F](\mathbf{x}) \end{aligned} \tag{A3}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$. The operator $V = -\delta a^k U_k^\mu \partial_\mu \equiv \delta a^k P_k$ is an infinitesimal isovector in the geometric language.

Infinitesimal group operators P_k $(k = 1, \ldots, n)$ are defined by

$$\begin{aligned}[P_k F](\mathbf{x}) &\equiv \frac{\partial}{\partial a^k}[T(g)F](\mathbf{x})|\, a^1 = \cdots = a^n = 0 \\ &= [-U_k^\mu(\mathbf{x})\partial_\mu F](\mathbf{x}). \end{aligned} \tag{A4}$$

The operators $T(g)$ and P_k constitute realizations of G and its Lie algebra, respectively.

Consider now the case where G is a representation of a linear Lie group and assume that, infinitesimally,

$$\mathbf{x}' = g\mathbf{x} \simeq (1 + \delta a^k L_k)\mathbf{x}, \tag{A5}$$

where the matrices L_k $(k = 1, \ldots, n)$ satisfy the usual commutation relations

$$[L_i, L_j] = C_{ij}^k L_k.$$

The operators P_k of Equation (A4) are now written

$$P_k = -(L_k)_\nu^\mu x^\nu \partial_\mu \tag{A6}$$

and they satisfy $[P_i, P_j] = C_{ij}^k P_k$.

In particular, if $\mathbf{x}$ is a vector transforming according to the adjoint representation of G: $(L_i)_j^k = C_{ij}^k$, then Equations (A5) and (A6) are written, respectively,

$$x^{k'} \simeq x^k + C_{ij}^k \delta a^i x^j, \tag{A5$'$}$$

$$P_i = -C_{ij}^k x^j \partial_k. \tag{A6$'$}$$

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8. To see this, write $V = V^k\partial/\partial z^k$ $(k = 1, \ldots, n')$ and use the fact that, for infinitesimal values of the group parameters, $-V^k$ represents a small change in the coordinate z^k (see Appendix).
9. In this paragraph we adopt the following notation for unit matrices: 1_4 and 1_n denote the 4×4 and $n\times n$ unit matrices, respectively, and 1 denotes the $4n\times 4n$ unit matrix $1_n 1_4$ (tensor product is assumed).
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Some aspects of the isogroup of the self-dual Yang–Mills system

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A generalized isovector formalism is used to derive the isovectors and isogroup of the self-dual Yang–Mills (SDYM) equation in the so-called J formulation. In particular, the infinitesimal "hidden symmetry" transformation, a linear system, and a well-known Bäcklund transformation of the SDYM equation are derived in the process. Thus symmetry and integrability aspects of the SDYM system appear in natural relationship to each other within the framework of the isovector approach.

I. INTRODUCTION

In a recent paper[1] the authors discussed the application of isovector techniques[2,3] to systems of partial differential equations corresponding to exterior equations for vector-valued (and, in particular, matrix-valued) differential forms. It was seen that the application of the Lie derivative operator on vector-valued one-forms presents some technical difficulties, and for this reason an *internal exterior derivative* (i.e., an exterior derivative that acts on the fields but not on the variables of the solution manifold) was introduced by the formula

$$\bar{d}F(x^\mu,\psi^i) \equiv dF - \partial_\mu F\, dx^\mu, \tag{1.1}$$

where F is any function of the scalar variables x^μ of the solution manifold and the vector-valued fields ψ^i. If the system of partial differential equations (PDE's) is of order 2 or higher, the variables ψ^i will comprise the dependent variables u^a of the PDE's and the derivatives, up to a certain degree, of the u^a with respect to the x^μ. Given that, in the absence of specific restrictions on the exterior differential forms that represent the system, the variables ψ^i are considered independent of each other (and of the x^μ), we conclude that the problem can be naturally formulated on a jet space with "mixed" (i.e., both scalar- and vector-valued) coordinates.

In the present paper the formalism developed in Ref. 1 is applied to the self-dual Yang–Mills (SDYM) equation in the so-called J formulation.[4] It is seen that the isovector method provides a natural framework for the unification of such distinct concepts as symmetry and integrability. The independence of the coordinates of the underlying jetlike space is important in this context, as the reader will realize. In Sec. II we calculate the isovectors of the SDYM system. These vector fields can be used to construct infinitesimal symmetries (both geometrical and internal) of the system, as discussed in Ref. 1. The above-mentioned independence of coordinates is used in Sec. III to rewrite certain symmetries in a form equivalent to the parametric infinitesimal transformation introduced in Ref. 5. (This transformation is related to the so-called hidden symmetry of the SDYM field.[6]) Remarkably, the process also yields a pair of linear "inverse scattering" equations, the integrability of which is equivalent to the SDYM equation, and the parameter of which is identical to that of the infinitesimal transformation mentioned above. Finally, the results of Secs. I–III are used in Sec. IV to derive Bäcklund transformations for the SDYM system. In particular, the process gives the parametric Bäcklund transformation proposed in Ref. 7.

II. ISOVECTORS OF THE SDYM SYSTEM

The SDYM equation in the J formulation is written as[4–7]

$$\partial_{\bar{y}}(J^{-1}\partial_y J) + \partial_{\bar{z}}(J^{-1}\partial_z J) = 0. \tag{2.1}$$

The complex coordinates y, z, $\bar{y}$, and $\bar{z}$ are related to the coordinates x_1, x_2, x_3, and x_4 of complexified Euclidean space by

$$\begin{aligned} 2^{1/2}y &= x_1 + ix_2, \quad 2^{1/2}z = x_3 - ix_4,\\ 2^{1/2}\bar{y} &= x_1 - ix_2, \quad 2^{1/2}\bar{z} = x_3 + ix_4. \end{aligned} \tag{2.2}$$

[Note that the pairs $(y,\bar{y})$ and $(z,\bar{z})$ involve elements that are complex-conjugately related in *real* Euclidean space.] For our purposes, J is assumed to be a nonsingular element of the algebra $\mathrm{gl}(N,C)$ in its defining representation.

Equation (2.1) can be rewritten as a set of first-order PDE's:

$$B^1_{\bar{y}} + B^2_{\bar{z}} = 0, \quad B^1 = J^{-1}J_y, \quad B^2 = J^{-1}J_z, \tag{2.3}$$

where a standard notation for partial derivatives has been used. We are thus led, in the spirit of Ref. 1, to define the following set of four-forms in seven variables:

$$\begin{aligned} \gamma_1 &= dy\, dz\, dB^1\, d\bar{z} + dy\, dz\, d\bar{y}\, dB^2,\\ \gamma_2 &= dJ\, dz\, d\bar{y}\, d\bar{z} - JB^1\, dy\, dz\, d\bar{y}\, d\bar{z},\\ \gamma_3 &= dy\, dJ\, d\bar{y}\, d\bar{z} - JB^2\, dy\, dz\, d\bar{y}\, d\bar{z}. \end{aligned} \tag{2.4}$$

It is easily seen that the $d\gamma_k$ are in the ideal of the γ_k; thus this ideal is closed.

We now proceed to find the isovectors of the system. For this purpose we must expand the Lie derivative of each γ_i into a "linear" combination of all three γ_k. The expansion must be made consistently with the requirement that the Lie derivative preserve the tensorial character of each γ_i separately.

Now, from Eqs. (2.4) it can be seen that the four-forms γ_k have values in $\mathrm{gl}(N,C)$, which is closed under both addition and multiplication. This observation suggests the following expansion:

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$$\underset{V}{\pounds}\gamma_i = b_i^{\,k}\gamma_k + \Lambda_i^k\gamma_k + \gamma_k M_i^{\,k}, \tag{2.5}$$

where the $b_i^{\,k}$ are scalars, whereas the zero-forms Λ_i^k and $M_i^{\,k}$ have values in $\mathrm{gl}(N,C)$.

The vector field V is defined on a jetlike space with "coordinates" $y, z, \bar{y}, \bar{z}, J, B^1$, and B^2. As argued in Ref. 1, V will have a formal representation,

$$V = D^1\frac{\partial}{\partial y} + D^2\frac{\partial}{\partial z} + D^3\frac{\partial}{\partial \bar{y}} + D^4\frac{\partial}{\partial \bar{z}} + G\frac{\partial}{\partial J} + A^1\frac{\partial}{\partial B^1} + A^2\frac{\partial}{\partial B^2}, \tag{2.6}$$

where the $D^1,...,D^4$ are assumed to be scalar functions of $y, z, \bar{y}, \bar{z}$, while the G, A^1, A^2 are $\mathrm{gl}(N,C)$-valued functions of the above four variables and J, B^1, and B^2. As in Ref. 1, we seek vector fields V for which the coefficients of expansion in Eq. (2.5) depend only on y, z, $\bar{y}$, and $\bar{z}$.

Substituting Eqs. (2.4) and (2.6) into Eq. (2.5), and using Eq. (1.1) to write

$$\pounds\, dJ = dG = G_{,\mu}\, dy^\mu + \bar{d}G,$$
$$\pounds\, dB^k = dA^k = A^k_{,\mu}\, dy^\mu + \bar{d}A^k \quad (k=1,2),$$

where the y^μ $(\mu = 1,...,4)$ denote the $y,...,\bar{z}$, we obtain a set of three exterior equations for four-forms. By equating the coefficients of $dy\, dz\, d\bar{y}\, d\bar{z}$ on both sides of each exterior equation, the following set of PDE's is derived:

$$\begin{aligned}
&A^1_{\bar{y}} + A^2_{\bar{z}} = -(b_1^{\,2} + \Lambda_1^2)JB^1 - (b_1^{\,3} + \Lambda_1^3)JB^2 \\
&\qquad - JB^1M_1^{\,2} - JB^2M_1^{\,3}, \\
&G_y - GB^1 - JA^1 - D^\mu_{,\mu}JB^1 \\
&\quad = -(b_2^{\,2} + \Lambda_2^2)JB^1 - (b_2^{\,3} + \Lambda_2^3)JB^2 \\
&\quad\ - JB^1M_2^{\,2} - JB^2M_2^{\,3}, \\
&G_z - GB^2 - JA^2 - D^\mu_{,\mu}JB^2 \\
&\quad = -(b_3^{\,2} + \Lambda_3^2)JB^1 - (b_3^{\,3} + \Lambda_3^3)JB^2 \\
&\quad\ - JB^1M_3^{\,2} - JB^2M_3^{\,3},
\end{aligned} \tag{2.7}$$

where $D^\mu \equiv D^1,...,D^4$.

We now put

$$\begin{aligned}
A^i &= \alpha^{ik}(y^\mu)B^k + \beta^i(y^\mu)J + \bar{A}^i(y^\mu,B^k,J), \\
G &= \delta^k(y^\mu)B^k + \epsilon(y^\mu)J + \bar{G}(y^\mu,B^k,J),
\end{aligned} \tag{2.8}$$

where the α^{ik}, β^i, δ^k, and ϵ are scalars. Then

$$\bar{d}A^i = \alpha^{ik}\, dB^k + \beta^i\, dJ + \overline{d\bar{A}}^{\,i},$$
$$\bar{d}G = \delta^k\, dB^k + \epsilon\, dJ + \overline{d\bar{G}}.$$

We substitute these expressions into the expansion of Eq. (2.5) and equate coefficients of terms that are *scalar* multiples of similar $\mathrm{gl}(N,C)$-valued basis four-forms. There are 12 such basis four-forms; therefore we obtain a set of 36 equations (eight of which are trivial identities). These results can be summarized as follows:

$$\beta^1 = \beta^2 = \delta^1 = \delta^2 = 0; \quad \alpha^{12} = D^3_{\bar{z}}, \quad \alpha^{21} = D^4_{\bar{y}};$$
$$D^1_{\bar{y}} = D^1_{\bar{z}} = 0, \quad D^2_{\bar{y}} = D^2_{\bar{z}} = 0,$$
$$D^3_y = D^3_z = 0, \quad D^4_y = D^4_z = 0;$$
$$b_1^{\,1} = D^1_y + D^2_z + D^3_{\bar{y}} + \alpha^{22} = D^1_y + D^2_z + D^4_{\bar{z}} + \alpha^{11},$$
$$b_2^{\,2} = D^2_z + D^3_{\bar{y}} + D^4_{\bar{z}} + \epsilon, \quad b_3^{\,3} = D^1_y + D^3_{\bar{y}} + D^4_{\bar{z}} + \epsilon,$$
$$b_2^{\,3} = -D^2_y, \quad b_3^{\,2} = -D^1_z, \quad b_1^{\,2} = b_1^{\,3} = b_2^{\,1} = b_3^{\,1} = 0.$$

We notice, in particular, that the D^1 and D^2 depend only on y and z, while the D^3 and D^4 depend only on $\bar{y}$ and $\bar{z}$.

The remaining terms in the expansion of Eq. (2.5) are those that cannot be expressed as *scalar* multiples of basis four-forms [in the sense that the coefficients in these terms do not commute with the $\mathrm{gl}(N,C)$-valued basis four-forms]. Terms of this type can be divided into four kinds according to their dependence on the basis three-forms $dy\, dz\, d\bar{y}$, $dy\, dz\, d\bar{z}$, $dy\, d\bar{y}\, d\bar{z}$, or $dz\, d\bar{y}\, d\bar{z}$. The $\mathrm{gl}(N,C)$-valued coefficients of each of these basis three-forms must be equated in each of the three exterior equations; this process yields a set of 12 equations which can be divided into two general types:

$$\Lambda_i^k\, dY + (dY)M_i^{\,k} = 0, \quad i \neq k, \tag{2.9}$$

and

$$\bar{d}\bar{H} = \Lambda_k^k\, dY + (dY)M_k^{\,k}, \tag{2.10}$$

where $Y \equiv B^1, B^2, J$ and $H \equiv A^1, A^2, G$. The variable Y, by assumption, does not commute with Λ_i^k and $M_i^{\,k}$. Thus Eq. (2.9) is satisfied only if $\Lambda_i^k = M_i^{\,k} = 0, i \neq k$. Also, given that, by definition of the internal exterior derivative and by assumption about the Λ_i^k and $M_i^{\,k}$, $dY = \bar{d}Y$, $\bar{d}\Lambda_k^k(y^\mu) = 0$, $\bar{d}M_k^{\,k}(y^\mu) = 0$, Eq. (2.10) can be integrated immediately:

$$\bar{H} = \Lambda_k^k Y + YM_k^{\,k} + h(y^\mu),$$

where $h(y^\mu)$ is an arbitrary function. Our results are explicitly stated as follows:

$$\begin{aligned}
&\Lambda_1^1 \equiv \Lambda^1(y^\mu), \quad M_1^{\,1} \equiv M^1(y^\mu), \\
&\Lambda_2^2 = \Lambda_3^3 \equiv \Lambda^2(y^\mu), \quad M_2^{\,2} = M_3^{\,3} \equiv M^2(y^\mu), \\
&\Lambda_i^k = M_i^{\,k} = 0, \quad \text{for } i \neq k; \\
&\bar{A}^1 = \Lambda^1B^1 + B^1M^1 + h^1(y^\mu), \\
&\bar{A}^2 = \Lambda^1B^2 + B^2M^1 + h^2(y^\mu), \\
&\bar{G} = \Lambda^2J + JM^2 + g(y^\mu),
\end{aligned}$$

where the h^1, h^2, and g are arbitrary $\mathrm{gl}(N,C)$-valued functions.

Appropriate substitutions into Eqs. (2.8) will now give expressions for A^i and G, which can be substituted back into Eqs. (2.7). By using previous results, the coefficients $b_i^{\,k}$ can be eliminated in favor of other quantities, while certain replacements can also be made with regard to the Λ_i^k and $M_i^{\,k}$. The result is a set of equalities between some kind of generalized "polynomial" expressions in the variables B^1, B^2, and J, with y^μ-dependent coefficients. The "constant" term in such a "polynomial" is a matrix function $F(y^\mu)$, while the other terms are of the following kinds: qB^k, qJ, qJB^k, QB^k, B^kQ, QJ, JQ, QJB^k, JQB^k, and JB^kQ, where $q(y^\mu)$ is a scalar function and $Q(y^\mu)$ is a $\mathrm{gl}(N,C)$-valued function. Equating coefficients of similar terms we obtain a set of partial differential and algebraic equations, which are not hard to solve. In particular, we find

$$-\Lambda^1 = M^1 = M^2 \equiv M(y,z), \quad \Lambda^2 \equiv \Lambda(\bar{y},\bar{z}),$$
$$h^1(y,z) = M_y, \quad h^2(y,z) = M_z, \quad g(y^\mu) = 0.$$

Equations (2.11) give the complete solution for the components of the isovector field V:

$$
\begin{aligned}
&D^1 = c_1 y + k_1 z + \alpha_1, \quad D^2 = k_2 y + c_2 z + \alpha_2,\\
&D^3 = (c_2 - c)\bar{y} - k_2 \bar{z} + \alpha_3,\\
&D^4 = -k_1 \bar{y} + (c_1 - c)\bar{z} + \alpha_4,\\
&A^1 = -c_1 B^1 - k_2 B^2 - [M(y,z),B^1] + M_y,\\
&A^2 = -k_1 B^1 - c_2 B^2 - [M(y,z),B^2] + M_z,\\
&G = \epsilon(\bar{y},\bar{z})J + \Lambda(\bar{y},\bar{z})J + JM(y,z),
\end{aligned}
\tag{2.11}
$$

where $c_1, c_2, k_1, k_2, c, \alpha_1,..., \alpha_4$ are nine complex parameters, $\epsilon(\bar{y},\bar{z})$ is a scalar function, and $M(y,z)$ and $\Lambda(\bar{y},\bar{z})$ are $\mathrm{gl}(N,C)$-valued functions. From Eqs. (2.11) we can read off the infinitesimal operators P_k corresponding to the nine complex parameters (cf. Ref. 1) and we can show that they form the basis of a Lie algebra. In particular, the operators

$$P_{\alpha_\mu} = \frac{\partial}{\partial y^\mu}$$

and

$$P_{c_1} + P_{c_2} = y^\mu \frac{\partial}{\partial y^\mu} - B^k \frac{\partial}{\partial B^k}$$

represent translations and dilatations, respectively.

Following the discussion in Ref. 1, from Eq. (2.11) we can construct the following infinitesimal internal symmetry transformations:

$$
\begin{aligned}
&B^{1\prime} \simeq B^1 + [M(y,z),B^1] - M_y,\\
&B^{2\prime} \simeq B^2 + [M(y,z),B^2] - M_z,\\
&J' \simeq J - \epsilon(\bar{y},\bar{z})J - \Lambda(\bar{y},\bar{z})J - JM(y,z),
\end{aligned}
\tag{2.12}
$$

where the ϵ, M, and Λ are infinitesimal. The corresponding finite transformations are

$$
\begin{aligned}
&B^{1\prime} = UB^1U^{-1} + U\partial_y U^{-1},\\
&B^{2\prime} = UB^2U^{-1} + U\partial_z U^{-1},\\
&J' = \beta\overline{U}\, J\, U,
\end{aligned}
\tag{2.13}
$$

where

$$
\begin{aligned}
&U(y,z) = \exp\{-M(y,z)\},\\
&\overline{U}(\bar{y},\bar{z}) = \exp\{-\Lambda(\bar{y},\bar{z})\},
\end{aligned}
$$

and

$$\beta(\bar{y},\bar{z}) = \exp\{-\epsilon(\bar{y},\bar{z})\}.$$

These are, of course, familiar symmetries of the SDYM system.

III. PARAMETRIC INFINITESIMAL TRANSFORMATION AND LINEAR SYSTEM

If we define a new function

$$\xi(y,z,\bar{y},\bar{z}) \equiv M(y,z) + \epsilon(\bar{y},\bar{z})1_N, \tag{3.1}$$

where 1_N denotes the N-dimensional unit matrix, then the infinitesimal transformations of Eq. (2.12) with $\Lambda(\bar{y},\bar{z}) = 0$ can be rewritten as

$$
\begin{aligned}
&\delta B^1 = [\xi(y^\mu),B^1] - \xi_y,\\
&\delta B^2 = [\xi(y^\mu),B^2] - \xi_z, \quad \delta J = -J\xi(y^\mu),
\end{aligned}
\tag{3.2}
$$

where $\delta B^k \simeq B^{k\prime} - B^k$ and $\delta J \simeq J' - J$. We wish to rewrite these symmetries without the restriction (3.1). It turns out that this is possible due to the independence of the coordinates of the underlying jetlike space. Of course, there is a price to be paid for such an adjustment. But this "price" is a most welcome one: Restriction (3.1) is replaced by a set of *linear* PDE's which, in the case of actual SDYM fields, lead to a linear system for the SDYM equation.

From Eq. (3.1) it is seen that $\xi(y^\mu)$ satisfies the PDE,

$$[\xi_{\bar{y}},B^1] + [\xi_{\bar{z}},B^2] - \xi_{y\bar{y}} - \xi_{z\bar{z}} = 0.$$

Given the independence of the y^μ and the B^k (this is the case as long as no restriction on the solution manifold is imposed), the above equation may be written as

$$\partial_{\bar{y}}([\xi,B^1] - \xi_y) + \partial_{\bar{z}}([\xi,B^2] - \xi_z) = 0. \tag{3.3}$$

This is satisfied if there exists a "potential" $\psi(y^\mu,B^k)$ such that

$$[\xi,B^1] - \xi_y = \lambda\psi_{\bar{z}}, \quad [\xi,B^2] - \xi_z = -\lambda\psi_{\bar{y}}, \tag{3.4}$$

where λ is an arbitrary complex parameter. We thus replace the system of Eqs. (3.1) and (3.2) by the following alternate one:

$$\delta B^1 = \lambda\psi_{\bar{z}}, \quad \delta B^2 = -\lambda\psi_{\bar{y}}, \quad \delta J = -J\xi(y^\mu), \tag{3.5}$$

where ψ and ξ satisfy the linear system (3.4). Note that Eqs. (3.4) and (3.5) become independent of Eqs. (3.1) and (3.2) upon restriction to the solution manifold, i.e., for actual SDYM fields.

Let us explore further the significance of Eqs. (3.4) for *actual* SDYM fields (in which case the B^k are *dependent upon* the y^μ). In particular, let us examine the ansatz $\psi(y^\mu) = \xi(y^\mu)$, all y^μ:

$$[\xi,B^1] - \xi_y = \lambda\xi_{\bar{z}}, \quad [\xi,B^2] - \xi_z = -\lambda\xi_{\bar{y}}. \tag{3.6}$$

The integrability criterion $\xi_{\bar{y}\bar{z}} - \xi_{\bar{z}\bar{y}} = 0$ yields Eq. (3.3), which, in combination with Eq. (3.6), gives

$$[\xi, B^2_y - B^1_z + [B^1,B^2] + \lambda(B^1_{\bar{y}} + B^2_{\bar{z}})] = 0.$$

We seek conditions for B^1 and B^2 in order that the above equality holds for all λ and independently of ξ. The following pair of PDE's must therefore be satisfied:

$$\partial_y B^2 - \partial_z B^1 + [B^1,B^2] = 0, \tag{3.7}$$

$$\partial_{\bar{y}} B^1 + \partial_{\bar{z}} B^2 = 0. \tag{3.8}$$

Equation (3.7) is a condition for zero curvature and implies that the B^1 and B^2 are pure gauges:

$$B^1 = J^{-1}\partial_y J, \quad B^2 = J^{-1}\partial_z J, \tag{3.9}$$

where J is a nonsingular $\mathrm{gl}(N,C)$ matrix. Then Eq. (3.8) becomes identical to the SDYM equation (2.1), of which Eq. (3.6) is seen to be a linear system.

We remark that our results are in agreement with those of Ref. 5 (although they are given in a slightly different form). The thing to notice is that these results were actually *derived* here, in a rather straightforward manner, by using the isovector technique.

IV. CONNECTION WITH BÄCKLUND TRANSFORMATIONS

By using the original definitions of B^1 and B^2 as given in Eqs. (2.3), the infinitesimal transformations of these quantities may be written, according to Eqs. (3.5) as

$$J'^{-1}J'_y - J^{-1}J_y = \lambda\psi_{\bar z}, \qquad (4.1a)$$

$$J'^{-1}J'_z - J^{-1}J_z = -\lambda\psi_{\bar y}. \qquad (4.1b)$$

Clearly, as J' approaches J, the $\psi_{\bar y}$ and $\psi_{\bar z}$ must approach zero. One way to achieve this is to put

$$\psi = \xi = 1 - J^{-1}J'. \qquad (4.2)$$

Now, if the left-hand sides of Eqs. (4.1a) and (4.1b) are considered as *finite*, rather than infinitesimal differences, then Eqs. (4.1) and (4.2) constitute one possible form of the Bäcklund transformation (BT) proposed in Ref. 7. Alternatively, the infinitesimal parametric transformation (4.1) and (4.2) is also an infinitesimal BT. This was observed in Ref. 5, but we include it in the present discussion due to its direct (and quite interesting) relevance to the isovector method.

Incidentally, the transformation (3.1) and (3.2) is also an infinitesimal BT, with Eq. (3.1) being a sort of algebraic constraint. Indeed, putting $\xi = 1 - J^{-1}J'$ and introducing an arbitrary complex parameter μ, we write

$$J'^{-1}J'_y - J^{-1}J_y = \mu\{[J^{-1}J', J^{-1}J_y] - \partial_y(J^{-1}J')\}, \qquad (4.3a)$$

$$J'^{-1}J'_z - J^{-1}J_z = \mu\{[J^{-1}J', J^{-1}J_z] - \partial_z(J^{-1}J')\}, \qquad (4.3b)$$

$$J^{-1}J' = M(y,z) + \epsilon(\bar y,\bar z)1_N, \qquad (4.3c)$$

where $M(y,z)$ is $\mathrm{gl}(N,C)$ valued and $\epsilon(\bar y,\bar z)$ is a scalar. Taking $\partial_{\bar y}$(4.3a) + $\partial_{\bar z}$(4.3b) and using (4.3c), we find

$$\begin{aligned}&\{\partial_{\bar y}(J'^{-1}J'_y) + \partial_{\bar z}(J'^{-1}J'_z)\}\\ &\quad - \{\partial_{\bar y}(J^{-1}J_y) + \partial_{\bar z}(J^{-1}J_z)\}\\ &\quad = \mu[J^{-1}J', \partial_{\bar y}(J^{-1}J_y) + \partial_{\bar z}(J^{-1}J_z)],\end{aligned}$$

according to which J' satisfies the SDYM equation (2.1) if J does. Note that the BT was constructed so as to yield the trivial solution $J' = J$ as a particular solution [this corresponds to $M = 0$ and $\epsilon = 1$ in the algebraic constraint (4.3c)].

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Nonlocal symmetries and Bäcklund transformations for the self-dual Yang–Mills system

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The observation is made that generalized evolutionary isovectors of the self-dual Yang–Mills equation, obtained by "verticalization" of the geometrical isovectors derived in a previous paper [J. Math. Phys. **28**, 1261 (1987)], generate Bäcklund transformations for the self-dual system. In particular, new Bäcklund transformations are obtained by "verticalizing" the generators of point transformations on the solution manifold. A geometric ansatz for the derivation of such (generally nonlocal) symmetries is proposed.

I. INTRODUCTION

In previous papers[1,2] the authors have discussed isovector[3] techniques for partial differential equations (PDE's) associated with vector-valued differential forms. It was mentioned[2] that such a PDE (or system of PDE's) defines, through its solutions, sections of a vector bundle over the solution manifold. This manifold serves as a base space, while the fibers are isomorphic to some vector space (or Lie algebra).

In Ref. 1 the isovector approach was employed to derive point symmetries for the self-dual Yang–Mills (SDYM) equation in its so-called J formulation[4] (this is mathematically different from the usual formulation in which the self-duality condition is directly written in covariant form). The system was represented by three $\mathrm{gl}(N,C)$-valued four-forms in seven variables. Since these forms generated a differential ideal by themselves, we did not include the integrability condition of the system in the ideal (indeed, such an inclusion can be seen to be superfluous for the purpose of deriving *point* transformations).

Calculation of the isovectors gave a nine parameter group of transformations on the base space, together with a set of infinitesimal internal transformations in the fiber space in which the SDYM fields have values. It was then observed that the internal symmetries were related to parametric Bäcklund transformations (BT's) for the SDYM equation. In particular, a well-known[5,6] BT was recovered.

With the observation that internal symmetries are generated by *evolutionary*[7] (i.e., "vertical") vector fields (EVF's), it is natural to inquire for other EVF's that may generate BT's for the SDYM system. Such fields cannot be sought, of course, among the "geometrical" symmetries found in Ref. 1. The most accessible nongeometrical (i.e., nonlocal) symmetries at our disposal are those generated by the *evolutionary representatives*[7] of the nine generators of coordinate transformations mentioned previously. In Sec. III we establish the generalized isovector property of these EVF's by examining the effect of the corresponding Lie derivatives on the original ideal of the three four-forms. It is found that these Lie derivatives map this ideal into a larger ideal comprising the original system, its integrability conditions, and certain prolongations[8] of all of the above. The emergence of a prolonged ideal was to be expected since we are now dealing with Lie–Bäcklund-type symmetries. On the other hand, the appearance of the integrability conditions as an inseparable part of the system is quite interesting, considering the passive role these conditions played in the derivation of point symmetries.

In Sec. IV we construct the infinitesimal parametric nonlocal transformations generated by the aforementioned nine EVF's. It is seen that, by letting the transformation parameters be considered finite, rather than infinitesimal, the transformations of the prolongation[8] variables become BT's for the SDYM equation. This observation constitutes a further indication of the intimate connection between symmetry and integrability aspects[9] of nonlinear systems, and, in particular, of the SDYM system.[1,10]

To make the paper as self-contained as possible, we review in the next section some of the results of Ref. 1 that will be needed for the present treatment.

II. GEOMETRICAL SYMMETRIES OF THE SDYM SYSTEM

As in Ref. 1, we write the SDYM equation (a second-order nonlinear PDE) as a set of first-order PDE's:

$$B^1_{\bar{y}} + B^2_{\bar{z}} = 0, \tag{2.1a}$$

$$B^1 = J^{-1} J_y, \tag{2.1b}$$

$$B^2 = J^{-1} J_z, \tag{2.1c}$$

where the subscripts denote partial differentiation (partial derivatives will occasionally be used). The $y, z, \bar{y}, \bar{z}$, collectively denoted by x^μ ($\mu = 1,2,3,4$), are four complex coordinates,[1,4,5] while J is assumed to have values in $\mathrm{gl}(N,C)$. Loosely speaking, the B^1 and B^2 are "prolongation variables" for the system. The integrability condition $J_{yz} = J_{zy}$ yields

$$B^2_y - B^1_z + [B^1, B^2] = 0. \tag{2.2}$$

The system (2.1) can be represented by a set of three $\mathrm{gl}(N,C)$-valued four-forms:

$$\begin{aligned} \gamma_1 &= dy\, dz\, dB^1\, d\bar{z} + dy\, dz\, d\bar{y}\, dB^2, \\ \gamma_2 &= dJ\, dz\, d\bar{y}\, d\bar{z} - JB^1\, dy\, dz\, d\bar{y}\, d\bar{z}, \\ \gamma_3 &= dy\, dJ\, d\bar{y}\, d\bar{z} - JB^2\, dy\, dz\, d\bar{y}\, d\bar{z}. \end{aligned} \tag{2.3}$$

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These forms generate a differential ideal. Indeed,

$$d\gamma_1 = 0, \quad d\gamma_2 = \gamma_2\, dy\, B^1 + J\, d\bar{y}\, \gamma_1,$$

$$d\gamma_3 = \gamma_3\, dz\, B^2 + J\, d\bar{z}\, \gamma_1.$$

The geometrical (local) symmetries of the system are generated by vector fields of the form[1]

$$V = \xi^\mu(x^\nu)\frac{\partial}{\partial x^\mu} + G(x^\nu,J)\frac{\partial}{\partial J} + A^i(x^\nu,B^k)\frac{\partial}{\partial B^i}, \qquad (2.4)$$

where the usual summation convention is assumed over repeated indices. The ξ^μ ($\mu = 1,...,4$) are scalars, whereas the G and A^i ($i = 1,2$) are gl(N,C) valued. The $\partial/\partial J$ and $\partial/\partial B^i$ are *formal* operators only—i.e., no differentiations are actually performed. The symmetry property is expressed by the requirement that the Lie derivative with respect to V leave the ideal of the γ_k invariant. Formally,

$$\underset{V}{\pounds}\gamma_i = b_i^k\gamma_k + \Lambda_i^k\gamma_k + \gamma_k M_i^k \qquad (2.5)$$

($i = 1,2,3$), where the b_i^k are scalars, whereas the Λ_i^k and M_i^k are gl(N,C)-valued zero-forms. The calculation of V from Eq. (2.5) becomes possible if we make the ansatz that the coefficients of expansion in this equation depend only on the x^μ. As we will see shortly, this condition is violated in the case of nongeometrical symmetries.

As was seen in Ref. 1, the vector V is parametrized by nine (complex) parameters, and depends on three arbitrary functions. The nine parameters correspond to transformations on the base space. These transformations are generated by the following independent vector fields, which are written here in unprolonged form (i.e., without the terms in $\partial/\partial B^i$):

$$V_\mu = -\frac{\partial}{\partial x^\mu}, \quad \mu = 1,2,3,4, \quad V_5 = -y\frac{\partial}{\partial y} - \bar{z}\frac{\partial}{\partial \bar{z}},$$

$$V_6 = -z\frac{\partial}{\partial z} - \bar{y}\frac{\partial}{\partial \bar{y}}, \quad V_7 = -z\frac{\partial}{\partial y} + \bar{y}\frac{\partial}{\partial \bar{z}}, \qquad (2.6)$$

$$V_8 = -y\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{y}}, \quad V_9 = -\bar{y}\frac{\partial}{\partial \bar{y}} - \bar{z}\frac{\partial}{\partial \bar{z}}.$$

Internal symmetries are generated by EVF's with components

$$G = \epsilon(\bar{y},\bar{z})J + \Lambda(\bar{y},\bar{z})J + JM(y,z),$$

$$A^1 = -[M(y,z),B^1] + M_y,$$

$$A^2 = -[M(y,z),B^2] + M_z,$$

where ϵ is a scalar function, while Λ and M are gl(N,C)-valued functions. The symmetries in ϵ and M combine to give BT's for SDYM.[1] The symmetry in Λ yields a BT that is less interesting, since it merely consists of $\delta(J^{-1}J_y) = 0$, $\delta(J^{-1}J_z) = 0$ (i.e., $J'^{-1}J'_y = J^{-1}J_y$, etc.).

III. EVOLUTIONARY ISOVECTORS FOR FIRST-ORDER GENERALIZED SYMMETRIES

A. Nongeometrical vectors and prolongation forms

We now relax the geometrical requirement and seek generalized[7] symmetries of SDYM. We confine our attention to first-order symmetries generated by *evolutionary* vector fields (i.e., vector fields with vanishing projection on the base space) of the form

$$V = Q\frac{\partial}{\partial J} + R^1\frac{\partial}{\partial B^1} + R^2\frac{\partial}{\partial B^2}, \qquad (3.1)$$

where Q may depend on x^μ, J, and $J_\mu \equiv \partial_\mu J$. With the definitions

$$J_y = JB^1, \quad J_z = JB^2, \qquad (3.2a)$$

$$J_{\bar{y}} = E^1, \quad J_{\bar{z}} = E^2, \qquad (3.2b)$$

we have that

$$Q = Q(x^\mu,J,B^1,B^2,E^1,E^2). \qquad (3.3)$$

[The reader may be concerned with the appearance of several noncommuting variables in the functional dependence of Q. However, as Q is calculated (see below), no ambiguity in the order of these variables arises.]

It is easily seen [see Eqs. (3.11) and (3.12)] that the R^1 and R^2 depend, collectively, on the additional variables $B^1_\mu \equiv \partial_\mu B^1$ and $B^2_\mu \equiv \partial_\mu B^2$. The variables $B^2_{\bar{z}}$ and B^1_z can be eliminated from the problem by using the field equation (2.1a) and the integrability condition (2.2), respectively:

$$B^2_{\bar{z}} = -B^1_{\bar{y}}, \qquad (3.4)$$

$$B^1_z = B^2_y + [B^1,B^2]. \qquad (3.5)$$

Thus we are left with the variables

$$B^1_y = C^1, \quad B^1_{\bar{y}} = C^2, \quad B^1_{\bar{z}} = C^3,$$
$$B^2_y = C^4, \quad B^2_z = C^5, \quad B^2_{\bar{y}} = C^6. \qquad (3.6)$$

Equations (3.2) and (3.6) each admit six integrability conditions, thus a total of 12 such conditions can be written [including the one given by Eq. (2.2)].

Let us consider the basic system (2.1), together with its integrability condition (2.2). We prolong Eqs. (2.1a) and (2.2) in the usual way by taking the derivatives with respect to the x^μ. By convention, only those prolongations defined *within the variables at our disposal* are considered. Specifically, we can construct the y, z, and $\bar{y}$ prolongations of Eq. (2.1a), and the y and $\bar{y}$ prolongations of Eq. (2.2):

$$C^1_{\bar{y}} + C^4_{\bar{z}} = 0, \qquad (3.7a)$$

$$C^2_z + C^5_{\bar{z}} = 0, \qquad (3.7b)$$

$$C^2_{\bar{y}} + C^6_{\bar{z}} = 0; \qquad (3.7c)$$

$$C^4_y - C^1_z + [C^1,B^2] + [B^1,C^4] = 0, \qquad (3.8a)$$

$$C^6_y - C^2_z + [C^2,B^2] + [B^1,C^6] = 0. \qquad (3.8b)$$

[Note that, in a sense, the prolongations of Eqs. (2.1b) and (2.1c) are contained in Eq. (3.6).]

We now express our equations in terms of differential forms. Thus we define 28 four-forms corresponding, successively, to Eqs. (2.1), (2.2), (3.7), (3.8), (3.6) and its integrability conditions, and Eq. (3.2b), and the five remaining integrability conditions of (3.2) (we put $\tilde{\omega} \equiv dy\, dz\, d\bar{y}\, d\bar{z}$):

$$\gamma_1 = dy\, dz\, dB^1\, d\bar{z} + dy\, dz\, d\bar{y}\, dB^2,$$

$$\gamma_2 = dJ\, dz\, d\bar{y}\, d\bar{z} - JB^1\tilde{\omega},$$

$$\gamma_3 = dy\, dJ\, d\bar{y}\, d\bar{z} - JB^2\tilde{\omega},$$

$$\gamma_4 = dB^2\, dz\, d\bar{y}\, d\bar{z} - dy\, dB^1\, d\bar{y}\, d\bar{z} + [B^1,B^2]\tilde{\omega},$$

$$\gamma_5 = dy\, dz\, dC^1\, d\bar{z} + dy\, dz\, d\bar{y}\, dC^4,$$

$$\gamma_6 = dy\, dC^2\, d\bar{y}\, d\bar{z} + dy\, dz\, d\bar{y}\, dC^5,$$

$$\gamma_7 = dy\, dz\, dC^2\, d\bar{z} + dy\, dz\, d\bar{y}\, dC^6,$$
$$\gamma_8 = dC^4\, dz\, d\bar{y}\, d\bar{z} - dy\, dC^1\, d\bar{y}\, d\bar{z} + ([C^1,B^2] + [B^1,C^4])\widetilde{\omega},$$
$$\gamma_9 = dC^6\, dz\, d\bar{y}\, d\bar{z} - dy\, dC^2\, d\bar{y}\, d\bar{z} + ([C^2,B^2] + [B^1,C^6])\widetilde{\omega},$$
$$\gamma_{10} = dB^1\, dz\, d\bar{y}\, d\bar{z} - C^1\widetilde{\omega},$$
$$\gamma_{11} = dy\, dz\, dB^1\, d\bar{z} - C^2\widetilde{\omega},$$
$$\gamma_{12} = dy\, dz\, d\bar{y}\, dB^1 - C^3\widetilde{\omega},$$
$$\gamma_{13} = dB^2\, dz\, d\bar{y}\, d\bar{z} - C^4\widetilde{\omega},$$
$$\gamma_{14} = dy\, dB^2\, d\bar{y}\, d\bar{z} - C^5\widetilde{\omega}, \tag{3.9}$$
$$\gamma_{15} = dy\, dz\, dB^2\, d\bar{z} - C^6\widetilde{\omega},$$
$$\gamma_{16} = dC^2\, dz\, d\bar{y}\, d\bar{z} - dy\, dz\, dC^1\, d\bar{z},$$
$$\gamma_{17} = dC^3\, dz\, d\bar{y}\, d\bar{z} - dy\, dz\, d\bar{y}\, dC^1,$$
$$\gamma_{18} = dy\, dz\, dC^3\, d\bar{z} - dy\, dz\, d\bar{y}\, dC^2,$$
$$\gamma_{19} = dC^5\, dz\, d\bar{y}\, d\bar{z} - dy\, dC^4\, d\bar{y}\, d\bar{z},$$
$$\gamma_{20} = dC^6\, dz\, d\bar{y}\, d\bar{z} - dy\, dz\, dC^4\, d\bar{z},$$
$$\gamma_{21} = dy\, dC^6\, d\bar{y}\, d\bar{z} - dy\, dz\, dC^5\, d\bar{z},$$
$$\gamma_{22} = dy\, dz\, dJ\, d\bar{z} - E^1\widetilde{\omega},$$
$$\gamma_{23} = dy\, dz\, d\bar{y}\, dJ - E^2\widetilde{\omega},$$
$$\gamma_{24} = dy\, dz\, dE^2\, d\bar{z} - dy\, dz\, d\bar{y}\, dE^1,$$
$$\gamma_{25} = dE^1\, dz\, d\bar{y}\, d\bar{z} - dy\, dz\, d(JB^1)\, d\bar{z},$$
$$\gamma_{26} = dE^2\, dz\, d\bar{y}\, d\bar{z} - dy\, dz\, d\bar{y}\, d(JB^1),$$
$$\gamma_{27} = dy\, dE^1\, d\bar{y}\, d\bar{z} - dy\, dz\, d(JB^2)\, d\bar{z},$$
$$\gamma_{28} = dy\, dE^2\, d\bar{y}\, d\bar{z} - dy\, dz\, d\bar{y}\, d(JB^2)\,.$$

B. Generalized isovectors

The postulate (2.5), used to derive symmetries of the system, can now be generalized[11] by requiring that the Lie derivative with respect to a generalized EVF of the form (3.1) map the original ideal $\{\gamma_1,\gamma_2,\gamma_3\}$ into the prolonged ideal $\{\gamma_1,...,\gamma_{28}\}$. [We can see that such a generalization is necessary because the Lie derivative of the γ_i ($i = 1,2,3$) will yield variables which appear only in the prolonged forms γ_α.] Formally,

$$\underset{V}{\pounds}\, \gamma_i = b_i^\alpha \gamma_\alpha + \Lambda_i^\alpha \gamma_\alpha + \gamma_\alpha M_i^\alpha, \tag{3.10}$$

where $i = 1,2,3$, as before, but now the index α runs from 1 to 28. The coefficients Λ_i^α and M_i^α are no longer required to be independent of the internal variables J, B^1, and B^2.

One would like, of course, to solve Eq. (3.10) for the components Q, R^1, and R^2 of V and thus obtain independent first-order generalized symmetries of SDYM. The computations, however, are now of great complexity due to the presence of a very large set of variables and forms. Rather than solving Eq. (3.10) directly, we will instead construct certain solutions by utilizing the coordinate (point) symmetries of SDYM, as these are expressed by the vector fields of Eq. (2.6).

First of all, it is seen from Eqs. (3.9) and (3.10) that some of the forms of the prolonged ideal will not occur in the expansion of the Lie derivative in Eq. (3.10). Indeed, by comparing similar terms and taking into account Eq. (3.3), it can be shown that the forms γ_8, γ_{16}, γ_{17}, γ_{19}, and γ_{21} make no contribution and thus can be eliminated. This leaves us with a somewhat smaller ideal of 23 forms.

Second, since the B^1 and B^2 are prolongation variables, the R^1 and R^2 of Eq. (3.1) are expressible in terms of Q. Again, this can be done by comparing similar terms in Eq. (3.10), using the "internal exterior derivative" (introduced in Refs. 1 and 2) whenever necessary.

There is, however, an easier way to do this: Let us recall that the vector field V, in the form (3.1), defines an infinitesimal "motion" in the space of J, B^1, and B^2. Thus, if δt is an infinitesimal parameter, $\delta J = Q\delta t$. Furthermore, if we regard t as a variable parametrizing an integral curve of V, then $Q = \delta J/\delta t$ and

$$R^1 = \frac{\delta B^1}{\delta t} = -J^{-1}\frac{\delta J}{\delta t}J^{-1}D_y J + J^{-1}\frac{\delta}{\delta t}(D_y J) = -J^{-1}QB^1 + J^{-1}D_y Q \tag{3.11}$$

and similarly

$$R^2 = \frac{\delta B^2}{\delta t} = -J^{-1}QB^2 + J^{-1}D_z Q, \tag{3.12}$$

where the D_y and D_z are total derivatives.[12] [In general, total derivatives must be defined consistently with Eqs. (3.2) and (3.4)–(3.6). Thus, on functions of the form (3.3),

$$D_y = \frac{\partial}{\partial y} + JB^1\frac{\partial}{\partial J} + C^1\frac{\partial}{\partial B^1} + C^4\frac{\partial}{\partial B^2} + (E^1B^1 + JC^2)\frac{\partial}{\partial E^1} + (E^2B^1 + JC^3)\frac{\partial}{\partial E^2},$$

$$D_z = \frac{\partial}{\partial z} + JB^2\frac{\partial}{\partial J} + (C^4 + [B^1,B^2])\frac{\partial}{\partial B^1} + C^5\frac{\partial}{\partial B^2} + (E^1B^2 + JC^6)\frac{\partial}{\partial E^1} + (E^2B^2 - JC^2)\frac{\partial}{\partial E^2}.$$

Note that D_μ, like ∂_μ, is a *derivation*, i.e., it satisfies the Leibniz rule.]

Finally, we need to define the *evolutionary representative*[7] (or verticalization) of a "horizontal" vector field. The following definition is pertinent to the SDYM case but can be easily generalized: Given a vector field of the (unprolonged) form

$$V_h = \xi^\mu(x^\nu)\frac{\partial}{\partial x^\mu}, \tag{3.13}$$

one can construct an EVF (in prolonged form)

$$V_q = Q\frac{\partial}{\partial J} + R^1\frac{\partial}{\partial B^1} + R^2\frac{\partial}{\partial B^2} \tag{3.14}$$

such that

$$Q = -\xi^\mu D_\mu J, \tag{3.15}$$

where $D_y J = JB^1$, $D_z J = JB^2$, $D_{\bar{y}} J = E^1$, $D_{\bar{z}} J = E^2$, and where the components R^1 and R^2 are related to Q as in Eqs. (3.11) and (3.12). The nongeometrical vector field V_q is called the evolutionary representative of the geometrical field V_h.[13] The reason for this particular definition is that, if V_h is a symmetry, it can be shown[7] that V_q is also a symmetry.

We are now in a position to construct the (prolonged) evolutionary representatives of the nine vector fields given in Eq. (2.6). The symbol $\overline{V}_k$ will denote the representative of V_k:

$$\overline{V}_1 = JB^1 \frac{\partial}{\partial J} + C^1 \frac{\partial}{\partial B^1} + C^4 \frac{\partial}{\partial B^2},$$

$$\overline{V}_2 = JB^2 \frac{\partial}{\partial J} + (C^4 + [B^1,B^2]) \frac{\partial}{\partial B^1} + C^5 \frac{\partial}{\partial B^2},$$

$$\overline{V}_3 = E^1 \frac{\partial}{\partial J} + C^2 \frac{\partial}{\partial B^1} + C^6 \frac{\partial}{\partial B^2},$$

$$\overline{V}_4 = E^2 \frac{\partial}{\partial J} + C^3 \frac{\partial}{\partial B^1} - C^2 \frac{\partial}{\partial B^2},$$

$$\overline{V}_5 = (yJB^1 + \bar{z}E^2) \frac{\partial}{\partial J} + (B^1 + yC^1 + \bar{z}C^3) \frac{\partial}{\partial B^1} + (yC^4 - \bar{z}C^2) \frac{\partial}{\partial B^2},$$

$$\overline{V}_6 = (zJB^2 + \bar{y}E^1) \frac{\partial}{\partial J} + (zC^4 + z[B^1,B^2] + \bar{y}C^2) \times \frac{\partial}{\partial B^1} + (B^2 + zC^5 + \bar{y}C^6) \frac{\partial}{\partial B^2}, \tag{3.16}$$

$$\overline{V}_7 = (zJB^1 - \bar{y}E^2) \frac{\partial}{\partial J} + (zC^1 - \bar{y}C^3) \frac{\partial}{\partial B^1} + (B^1 + zC^4 + \bar{y}C^2) \frac{\partial}{\partial B^2},$$

$$\overline{V}_8 = (yJB^2 - \bar{z}E^1) \frac{\partial}{\partial J} + (B^2 + yC^4 + y[B^1,B^2] - \bar{z}C^2) \frac{\partial}{\partial B^1} + (yC^5 - \bar{z}C^6) \frac{\partial}{\partial B^2},$$

$$\overline{V}_9 = (\bar{y}E^1 + \bar{z}E^2) \frac{\partial}{\partial J} + (\bar{y}C^2 + \bar{z}C^3) \frac{\partial}{\partial B^1} + (\bar{y}C^6 - \bar{z}C^2) \frac{\partial}{\partial B^2}.$$

The consistency of these expressions with the geometrical derivation of symmetries, as this is expressed in the prescription (3.10), may now be seen. Direct substitution into Eq. (3.10) shows that the above EVF's are generalized isovectors for the SDYM system (details are deferred to Appendix A). Our search for other generalized symmetries, of the same order or higher, has not been successful so far. In particular, one can see that the (generalized) Lie brackets[7] of the known symmetries do not produce new symmetries, contrary to what one might have hoped (this point is more easily verified by using true jet-space variables x^μ, J, J_μ, and $J_{\mu\nu}$).

IV. INFINITESIMAL NONLOCAL SYMMETRIES AND BÄCKLUND TRANSFORMATIONS

By using the EVF's (3.16) and the definitions (3.2), the following infinitesimal parametric nonlocal transformations of the J function are constructed[7] (λ is an infinitesimal parameter):

$$\begin{aligned}
&\delta_\mu J = \lambda\, \partial_\mu J \equiv \lambda J_\mu,\ \mu = 1,2,3,4,\ \ \delta_5 J = \lambda (yJ_y + \bar{z}J_{\bar{z}}),\\
&\delta_6 J = \lambda (zJ_z + \bar{y}J_{\bar{y}}),\quad \delta_7 J = \lambda (zJ_y - \bar{y}J_{\bar{z}}),\\
&\delta_8 J = \lambda (yJ_z - \bar{z}J_{\bar{y}}),\quad \delta_9 J = \lambda (\bar{y}J_{\bar{y}} + \bar{z}J_{\bar{z}}).
\end{aligned} \tag{4.1}$$

We can also construct the corresponding transformations for the variables $B^1 = J^{-1}J_y$ and $B^2 = J^{-1}J_z$. We will initially regard these transformations as general nonlocal symmetries generated by the EVF's (3.16) and independent of the SDYM equation. This means that one is allowed to make replacements in the components of the EVF's according to the definitions (3.2) and (3.6) and the integrability condition (3.5), but one may not use the equation of motion (3.4) (thus the apparent asymmetry of the equations below). The transformation equations are

$$\begin{aligned}
&\delta_1(J^{-1}J_y) = \lambda\, \partial_y (J^{-1}J_y),\\
&\delta_1(J^{-1}J_z) = \lambda\, \partial_y (J^{-1}J_z),\\
&\delta_2(J^{-1}J_y) = \lambda\, \partial_z (J^{-1}J_y),\\
&\delta_2(J^{-1}J_z) = \lambda\, \partial_z (J^{-1}J_z),\\
&\delta_3(J^{-1}J_y) = \lambda\, \partial_{\bar{y}} (J^{-1}J_y),\\
&\delta_3(J^{-1}J_z) = \lambda\, \partial_{\bar{y}} (J^{-1}J_z),\\
&\delta_4(J^{-1}J_y) = \lambda\, \partial_{\bar{z}} (J^{-1}J_y),\\
&\delta_4(J^{-1}J_z) = -\lambda\, \partial_{\bar{y}} (J^{-1}J_y),\\
&\delta_5(J^{-1}J_y) = \lambda\, [J^{-1}J_y + y\, \partial_y (J^{-1}J_y) + \bar{z}\, \partial_{\bar{z}} (J^{-1}J_y)],\\
&\delta_5(J^{-1}J_z) = \lambda\, [y\, \partial_y (J^{-1}J_z) - \bar{z}\, \partial_{\bar{y}} (J^{-1}J_y)],\\
&\delta_6(J^{-1}J_y) = \lambda\, [z\, \partial_z (J^{-1}J_y) + \bar{y}\, \partial_{\bar{y}} (J^{-1}J_y)],\\
&\delta_6(J^{-1}J_z) = \lambda\, [J^{-1}J_z + z\, \partial_z (J^{-1}J_z) + \bar{y}\, \partial_{\bar{y}} (J^{-1}J_z)],\\
&\delta_7(J^{-1}J_y) = \lambda\, [z\, \partial_y (J^{-1}J_y) - \bar{y}\, \partial_{\bar{z}} (J^{-1}J_y)],\\
&\delta_7(J^{-1}J_z) = \lambda\, [J^{-1}J_y + z\, \partial_y (J^{-1}J_z) + \bar{y}\, \partial_{\bar{y}} (J^{-1}J_y)],\\
&\delta_8(J^{-1}J_y) = \lambda\, [J^{-1}J_z + y\, \partial_z (J^{-1}J_y) - \bar{z}\, \partial_{\bar{y}} (J^{-1}J_y)],\\
&\delta_8(J^{-1}J_z) = \lambda\, [y\, \partial_{\dot{z}} (J^{-1}J_z) - \bar{z}\, \partial_{\bar{y}} (J^{-1}J_z)],\\
&\delta_9(J^{-1}J_y) = \lambda\, [\bar{y}\, \partial_{\bar{y}} (J^{-1}J_y) + \bar{z}\, \partial_{\bar{z}} (J^{-1}J_y)],\\
&\delta_9(J^{-1}J_z) = \lambda\, [\bar{y}\, \partial_{\bar{y}} (J^{-1}J_z) - \bar{z}\, \partial_{\bar{y}} (J^{-1}J_y)].
\end{aligned} \tag{4.2}$$

We now observe that the above nonlocal transformations of the prolongation variables bear an interesting property: Suppose that we let the parameters λ become *finite* in each case (λ stands for nine different independent parameters). In this case the left-hand sides of Eq. (4.2) become *finite* differences:

$$\begin{aligned}
&\delta_k (J^{-1}J_y) \equiv J'^{-1}J'_y - J^{-1}J_y,\\
&\delta_k (J^{-1}J_z) \equiv J'^{-1}J'_z - J^{-1}J_z.
\end{aligned} \tag{4.3}$$

We thus obtain nine pairs of independent parametric equations. Cross-differentiation of each pair with respect to $\bar{y}$ and $\bar{z}$, and use of the various integrability conditions, will then reveal that all nine pairs are parametric Bäcklund transformations for the SDYM equation, which J is now required to satisfy. Specifically, if J is a solution of

$$\partial_{\bar{y}}(J^{-1}J_y)+\partial_{\bar{z}}(J^{-1}J_z)=0, \tag{4.4}$$

then so is J', which is related to J by Eqs. (4.2) and (4.3). A more detailed proof of this is given in Appendix B.

We remark that one could obtain a more symmetric set of equations than Eq. (4.2) by allowing the components of the EVF's (3.16) to be evaluated on solutions of the SDYM equation, i.e., by using Eq. (3.4) to reintroduce $B^2_{\bar{z}}$ into the problem. The reader is invited to construct this alternate set of BT's.

In order for the BT's described in Eqs. (4.2) and (4.3) to be valid, one must also require the integrability condition $(J'_y)_z=(J'_z)_y$. If L represents any one of the differential operators in Eq. (2.6) [which operators appear on the right-hand side of Eq. (4.2)], this condition can be shown to require $[H_y,H_z]=0$, where $H=(LJ)J^{-1}$. Thus this is a condition on the original solution J. Many solutions satisfy this condition, so that it is not excessively restrictive. This condition will be explored in future publications.

V. CONCLUSIONS AND SUMMARY

Let us summarize our main conclusions.

(1) The study of first-order Lie–Bäcklund type symmetries of the SDYM system requires the construction of a prolonged ideal of four-forms, which is many times larger than the ideal used for the derivation of point transformations. A noteworthy feature of the expanded ideal is the presence of the integrability conditions of the system.

(2) Starting with the (point) symmetries of SDYM on the base manifold, one can construct nine evolutionary vector fields that are generalized isovectors of the system. Nine nonlocal symmetries of SDYM are thus obtained. No further generalized symmetries can be generated by simply taking the Lie brackets of the nine EVF's.

(3) It is our conclusion that all evolutionary representatives of the corresponding point symmetries of SDYM yield Bäcklund transformations for the system. These BT's are "weak," in the sense that they relate two functions, of which the second is a solution of SDYM provided that the first one is. The physical implications of these transformations are not yet totally clear to us.

APPENDIX A: PROOF OF ISOVECTOR PROPERTY

We display the expansions of the Lie derivatives of the forms γ_1, γ_2, γ_3, with respect to the prolonged EVF's $\tilde{V}_s$ $(s=1,\dots,9)$ of Eq. (3.16). We will use the notation

$$\gamma_i^{(s)}\equiv\underset{\tilde{V}_s}{\pounds}\gamma_i \quad (i=1,2,3).$$

Analytically,

$$\gamma_1^{(1)}=\gamma_5,$$
$$\gamma_2^{(1)}=\gamma_2B^1+J\gamma_{10},$$
$$\gamma_3^{(1)}=\gamma_3B^1-J\gamma_4+J\gamma_{13},$$
$$\gamma_1^{(2)}=\gamma_6+\gamma_9+[\gamma_{11},B^2]+[B^1,\gamma_{15}]-\gamma_{20},$$
$$\gamma_2^{(2)}=\gamma_2B^2+J\gamma_{13},$$
$$\gamma_3^{(2)}=\gamma_3B^2+J\gamma_{14},$$
$$\gamma_1^{(3)}=\gamma_7,$$
$$\gamma_2^{(3)}=J\gamma_{11}+\gamma_{22}B^1+\gamma_{25},$$
$$\gamma_3^{(3)}=J\gamma_{15}+\gamma_{22}B^2+\gamma_{27},$$
$$\gamma_1^{(4)}=\gamma_{18},$$
$$\gamma_2^{(4)}=J\gamma_{12}+\gamma_{23}B^1+\gamma_{26},$$
$$\gamma_3^{(4)}=J\gamma_1-J\gamma_{11}+\gamma_{23}B^2+\gamma_{28},$$
$$\gamma_1^{(5)}=y\gamma_5+\gamma_{11}+\bar{z}\gamma_{18},$$
$$\gamma_2^{(5)}=\gamma_2yB^1+yJ\gamma_{10}+\bar{z}J\gamma_{12}+\gamma_{23}\bar{z}B^1+\bar{z}\gamma_{26},$$
$$\gamma_3^{(5)}=\bar{z}J\gamma_1+\gamma_3yB^1-yJ\gamma_4-\bar{z}J\gamma_{11}+yJ\gamma_{13}+\gamma_{23}\bar{z}B^2+\bar{z}\gamma_{28},$$
$$\gamma_1^{(6)}=\gamma_1+z\gamma_6+\bar{y}\gamma_7+z\gamma_9-\gamma_{11}+[\gamma_{11},zB^2]+[zB^1,\gamma_{15}]-z\gamma_{20},$$
$$\gamma_2^{(6)}=\gamma_2zB^2+\bar{y}J\gamma_{11}+zJ\gamma_{13}+\gamma_{22}\bar{y}B^1+\bar{y}\gamma_{25},$$
$$\gamma_3^{(6)}=\gamma_3zB^2+zJ\gamma_{14}+\bar{y}J\gamma_{15}+\gamma_{22}\bar{y}B^2+\bar{y}\gamma_{27},$$
$$\gamma_1^{(7)}=z\gamma_5+\gamma_{12}-\bar{y}\gamma_{18},$$
$$\gamma_2^{(7)}=\gamma_2zB^1+zJ\gamma_{10}-\bar{y}J\gamma_{12}-\gamma_{23}\bar{y}B^1-\bar{y}\gamma_{26},$$
$$\gamma_3^{(7)}=-\bar{y}J\gamma_1+\gamma_3zB^1-zJ\gamma_4+\bar{y}J\gamma_{11}+zJ\gamma_{13}-\gamma_{23}\bar{y}B^2-\bar{y}\gamma_{28},$$
$$\gamma_1^{(8)}=y\gamma_6-\bar{z}\gamma_7+y\gamma_9+[\gamma_{11},yB^2]+[yB^1,\gamma_{15}]+\gamma_{15}-y\gamma_{20},$$
$$\gamma_2^{(8)}=\gamma_2yB^2-\bar{z}J\gamma_{11}+yJ\gamma_{13}-\gamma_{22}\bar{z}B^1-\bar{z}\gamma_{25},$$
$$\gamma_3^{(8)}=\gamma_3yB^2+yJ\gamma_{14}-\bar{z}J\gamma_{15}-\gamma_{22}\bar{z}B^2-\bar{z}\gamma_{27},$$
$$\gamma_1^{(9)}=\bar{y}\gamma_7+\bar{z}\gamma_{18},$$
$$\gamma_2^{(9)}=\bar{y}J\gamma_{11}+\bar{z}J\gamma_{12}+\gamma_{22}\bar{y}B^1+\gamma_{23}\bar{z}B^1+\bar{y}\gamma_{25}+\bar{z}\gamma_{26},$$
$$\gamma_3^{(9)}=\bar{z}J\gamma_1-\bar{z}J\gamma_{11}+\bar{y}J\gamma_{15}+\gamma_{22}\bar{y}B^2+\gamma_{23}\bar{z}B^2+\bar{y}\gamma_{27}+\bar{z}\gamma_{28}.$$

APPENDIX B: PROOF OF BÄCKLUND TRANSFORMATIONS

Consider the nine pairs of parametric equations defined by Eqs. (4.2) and (4.3) (each pair share a common subscript k in δ_k). We show that each of these pairs is a (finite) BT for SDYM. For this purpose we cross-differentiate with respect to $\bar{y}$ and $\bar{z}$, and then add by terms, assuming that all integrability conditions are satisfied. We will use the notation

$$F[J]\equiv\partial_{\bar{y}}(J^{-1}J_y)+\partial_{\bar{z}}(J^{-1}J_z),$$

so that $F[J]=0$ implies the SDYM equation. From the nine pairs of equations we thus obtain, respectively,

$$F[J']-F[J]=\lambda\,\partial_yF[J],$$
$$F[J']-F[J]=\lambda\,\partial_zF[J],$$
$$F[J']-F[J]=\lambda\,\partial_{\bar{y}}F[J],$$
$$F[J']-F[J]=0,$$
$$F[J']-F[J]=\lambda y\,\partial_yF[J],$$

$$F[J']-F[J]=\lambda(1+z\,\partial_z+\bar{y}\,\partial_{\bar{y}})F[J],$$
$$F[J']-F[J]=\lambda z\,\partial_y F[J],$$
$$F[J']-F[J]=\lambda(y\,\partial_z-\bar{z}\,\partial_{\bar{y}})F[J],$$
$$F[J']-F[J]=\lambda\bar{y}\,\partial_{\bar{y}}F[J].$$

Thus $F[J]=0$ implies $F[J']=0$ in each case, which establishes the Bäcklund transformation property.

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[8]We emphasize that the term "prolongation" is used in a very liberal sense in this article, and it may apply to variables, vector fields, differential equations, or ideals. Its precise meaning will always be clear from the context.

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[11]This type of generalization can be supported by rigorous, general arguments that are not presented here. See, for example, P. H. M. Kersten, Ph.D. thesis, Twente University of Technology, The Netherlands, 1985.

[12]We write $B^1=J^{-1}D_yJ$ and $B^2=J^{-1}D_zJ$, where the x^μ and J are treated as independent variables at this stage (whence the use of total derivatives). Here, D_yJ and D_zJ are names for prolongation variables in the jet space, which reduce, upon projection to the solution manifold, simply to J_y and J_z.

[13]More generally, if $V=\xi^\mu\,\partial/\partial x^\mu+A\,\partial/\partial J$, then the evolutionary representative (3.14) of V is constructed by taking $Q=A-\xi^\mu D_\mu J$.

BÄCKLUND TRANSFORMATIONS AND LOCAL CONSERVATION LAWS FOR SELF-DUAL YANG–MILLS FIELDS WITH ARBITRARY GAUGE GROUP

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A process is described for deriving infinite sets of local conservation laws for self-dual Yang–Mills fields with arbitrary gauge group. This process utilizes a set of recently constructed Bäcklund transformations which leave the self-duality equation invariant.

1. Introduction

A set of nine parametric Bäcklund transformations (BTs) for the self-dual Yang–Mills (SDYM) equation were recently constructed [1] by using the coordinate symmetries of this equation [2]. The latter symmetries were expressed in the form of nonlocal transformations of the dependent variable, rather than point transformations in the coordinate space. It was subsequently observed [3] that this particular construction of BTs is possible due to the fact that the SDYM equation can be placed in the form of a conservation law.

An important discovery of a BT for SDYM was made shortly before the turn of the decade [4], but significant properties of this transformation, such as permutability and existence of a superposition formula, were only recently established [5]. In spite of these and other nice properties, this BT does not lend itself to the construction of an infinite number of local conservation laws (infinite sets of *nonlocal* conservation laws have been obtained by other means [6–8]).

In this Letter we propose to show that, due to their special form the BTs of ref. [1] lead to the derivation of infinite sequences of local conservation laws for SDYM.

The Letter is organized as follows:

In section 2, the BTs introduced in ref. [1] are presented and subsequently rewritten in compact matrix form. The permutability property then follows immediately. This property does not necessitate the existence of a nonlinear superposition formula, in contrast with the BT of refs. [4,5].

In section 3 the construction of many-parameter BTs is described. These BTs are used in section 4 to derive infinite collections of local conservation laws for SDYM.

2. Bäcklund transformations

We adopt the following form of the SDYM equation [9,10,4,6–8]:

$$\partial_{\bar{y}}(J^{-1}J_y)+\partial_{\bar{z}}(J^{-1}J_z)=0 \qquad (2.1)$$

(where $J_y=\partial_y J\equiv\partial J/\partial y$, etc.). The variables y, z, $\bar{y}$, $\bar{z}$, collectively denoted x^μ ($\mu=1$, 2, 3, 4, respectively) are constructed from the coordinates of an underlying complexified euclidean space in such a way that $\bar{y}$ and $\bar{z}$ become the complex conjugates of y and z, respectively, when the above space is real. Considering an arbitrary underlying gauge group, we assume that J is generally a complex, nonsingular, $N\times N$ matrix. (For SU(N) gauge theory the additional constraints $J^\dagger=J$ and $\det J=1$ must be satisfied in real space (the dagger denotes hermitian conjugation).)

We now write a set of nine parametric BTs [1,3] which, upon integration, produce "new" solutions J' of eq. (2.1) from "old" ones, J. We index each BT by $k=1, ..., 9$ and use the notation $A^1 \equiv J^{-1}J_y$, $A^2 \equiv J^{-1}J_z$, $\Delta_k A^1 \equiv J'^{-1}J'_y - J^{-1}J_y$, $\Delta_k A^2 \equiv J'^{-1}J'_z - J^{-1}J_z$. We denote by λ the complex parameter of each of these BTs:

$$\Delta_\mu A^1 = \lambda \partial_\mu A^1 \ , \quad \Delta_\mu A^2 = \lambda \partial_\mu A^2 \quad (\mu=1, 2, 3, 4) \ ;$$

$$\Delta_5 A^1 = \lambda[(1 + y\partial_y + \bar{z}\partial_{\bar{z}})A^1] \ ,$$

$$\Delta_5 A^2 = \lambda[(y\partial_y + \bar{z}\partial_{\bar{z}})A^2] \ ;$$

$$\Delta_6 A^1 = \lambda[(z\partial_z + \bar{y}\partial_{\bar{y}})A^1] \ ,$$

$$\Delta_6 A^2 = \lambda[(1 + z\partial_z + \bar{y}\partial_{\bar{y}})A^2] \ ;$$

$$\Delta_7 A^1 = \lambda[(z\partial_y - \bar{y}\partial_{\bar{z}})A^1] \ ,$$

$$\Delta_7 A^2 = \lambda[A^1 + (z\partial_y - \bar{y}\partial_{\bar{z}})A^2] \ ;$$

$$\Delta_8 A^1 = \lambda[(y\partial_z - \bar{z}\partial_{\bar{y}})A^1 + A^2] \ ,$$

$$\Delta_8 A^2 = \lambda[(y\partial_z - \bar{z}\partial_{\bar{y}})A^2] \ ;$$

$$\Delta_9 A^1 = \lambda[(\bar{y}\partial_{\bar{y}} + \bar{z}\partial_{\bar{z}})A^1] \ ,$$

$$\Delta_9 A^2 = \lambda[(\bar{y}\partial_{\bar{y}} + \bar{z}\partial_{\bar{z}})A^2] \ .$$

(The BT property of each pair of equations can be verified by cross-differentiation with respect to $\bar{y}$ and $\bar{z}$, followed by addition in order to satisfy eq. (2.1).)

In order that the above BTs produce new SDYM solutions J', the consistency condition $(J'_y)_z = (J'_z)_y$ must be satisfied (this will ensure that the pure gauges A^1 and A^2 transform into pure gauges $A^{1\prime} = J'^{-1}J'_y$ and $A^{2\prime} = J'^{-1}J'_z$). It can be checked that the above condition is always satisfied to the first order in λ (i.e., for λ infinitesimal). However, for a general, finite λ the integrability condition yields $[\Delta_k A^1, \Delta_k A^2] = 0$. This is an additional algebraic constraint on J that must be satisfied in order that the BTs be integrable for J'. As the purpose of this paper is the derivation of conservation laws (for any given known solution J), rather than the construction of new solutions J', we will not concentrate on this matter any further. Indeed, it will be seen that the integrability conditions have no effect on the validity of the conservation laws derived later. In what follows we will assume that the BTs are integrable, at least in principle. Thus we will continue to write quantities like $A^{1\prime}$ and $A^{2\prime}$ in pure-gauge form keeping in mind that these are formal expressions, in general.

Evidently, all nine BTs are of the general form

$$\begin{aligned} \mathbf{B}^\lambda(J^{-1}J_y) &\equiv J'^{-1}J'_y \\ &= J^{-1}J_y + \lambda[L_1(J^{-1}J_y) + L_2(J^{-1}J_z)] \ , \\ \mathbf{B}^\lambda(J^{-1}J_z) &\equiv J'^{-1}J'_z \\ &= J^{-1}J_z + \lambda[L_3(J^{-1}J_y) + L_4(J^{-1}J_z)] \ , \end{aligned} \tag{2.2}$$

where the $L_1, ..., L_4$ are local linear operators. Thus we have nine families of parametric equations of the form (2.2), the members of each family having the same set of operators $L_1, ..., L_4$ but different values of λ. We note that eq. (2.2) can be written in matrix form as follows:

$$\begin{bmatrix} J'^{-1}J'_y \\ J'^{-1}J'_z \end{bmatrix} = \mathbf{M}^\lambda \begin{bmatrix} J^{-1}J_y \\ J^{-1}J_z \end{bmatrix} , \tag{2.3}$$

where $\mathbf{M}^\lambda$ is the operator-valued matrix

$$\begin{bmatrix} 1+\lambda L_1 & \lambda L_2 \\ \lambda L_3 & 1+\lambda L_4 \end{bmatrix} .$$

By using the matrix form (2.3) of the BT $\mathbf{B}^\lambda$, the permutability property $\mathbf{B}^{\lambda_2}\mathbf{B}^{\lambda_1} = \mathbf{B}^{\lambda_1}\mathbf{B}^{\lambda_2}$ is readily verified. Note that the satisfaction of this property does not require additional algebraic constraints. Thus no nonlinear superposition formula for the above BT seems to exist (at least as a consequence of a permutability requirement).

3. Many-parameter transformations

We let $\Psi(J)$ denote the two-dimensional column vector with components $J^{-1}J_y$ and $J^{-1}J_z$. Double application of the BT $\mathbf{B}^\lambda$ on J, with different parameters λ_1 and λ_2, will the produce a new function K such that

$$\Psi(K) = \mathbf{M}^{\lambda_2}\mathbf{M}^{\lambda_1}\Psi(J) = \mathbf{M}^{\lambda_1}\mathbf{M}^{\lambda_2}\Psi(J) \ , \tag{3.1}$$

where

$$\mathbf{M}^{\lambda_2}\mathbf{M}^{\lambda_1} = \mathbf{M}^{\lambda_1+\lambda_2} + \lambda_1\lambda_2 \begin{bmatrix} L_1^2 + L_2L_3 & L_1L_2 + L_2L_4 \\ L_3L_1 + L_4L_3 & L_3L_2 + L_4^2 \end{bmatrix} .$$

From eq. (3.1) and the BT nature of $\mathbf{B}^{\lambda_1}$ and $\mathbf{B}^{\lambda_2}$ it follows that the SDYM property of J implies the same property for K. Thus the commutative "product" $\mathbf{B}^{\lambda_2}\mathbf{B}^{\lambda_1}$ is also a BT for SDYM, with matrix representation $\mathbf{M}^{\lambda_2}\mathbf{M}^{\lambda_1}$. Note that this new, two-parameter BT is *not* of the family $\mathbf{B}^{\lambda}$ for finite values of λ_1 and λ_2.

More generally, multiple application of $\mathbf{B}^{\lambda}$ on J, with different parameters $\lambda_1, ..., \lambda_n$, will produce a function K such that

$$\Psi(K)=\mathbf{M}^{\lambda_n}...\mathbf{M}^{\lambda_1}\Psi(J)\ , \tag{3.2}$$

where the matrix product on the right is commutative. Clearly, the "product" $\mathbf{B}^{\lambda_n}...\mathbf{B}^{\lambda_1}$ is an n-parameter BT for SDYM. The commutativity property indicates that only one such BT can be constructed from any given set of parameters.

It is worth noting that eq. (3.2) allows one to express $K^{-1}K_y$ and $K^{-1}K_z$ directly in terms of a known function J, and then, by a single integration, to find the new SDYM field K. In principle, this iterative process yields an infinity of SDYM solutions from any given one (provided, of course, that the appropriate integrability conditions are always satisfied).

4. Local conservation laws

Let us notice that eq. (2.1) has the form of a local continuity equation which is satisfied for all SDYM fields. An infinite number of such equations can be produced as follows.

We put $\lambda_1=...=\lambda_n\equiv\lambda$ in eq. (3.2) and write it as

$$\Psi(K)=(\mathbf{M}^{\lambda})^n\Psi(J)\ .$$

It follows from this that the quantities $K^{-1}K_y$ and $K^{-1}K_z$ are expansions in powers of λ:

$$K^{-1}K_y=\sum_{r=0}^{n}\lambda^r P_r\ ,\quad K^{-1}K_z=\sum_{r=o}^{n}\lambda^r Q_r\ . \tag{4.1}$$

Now, since K is an SDYM field, it satisfies the continuity equation $\partial_{\bar{y}}(K^{-1}K_y)+\partial_{\bar{z}}(K^{-1}K_z)=0$. Substituting from eq. (4.1), and equating coefficients of powers of λ to zero, we obtain the $(n+1)$ local continuity equations

$$\partial_{\bar{y}}P_r+\partial_{\bar{z}}Q_r=0\quad (r=0, 1, ..., n)\ . \tag{4.2}$$

By letting $n\to\infty$, we obtain an infinite set of local conservation laws from each family of BTs $\mathbf{B}^{\lambda}$.

The local property of these conservation laws follows from the observation that, for any value of r, the "densities" P_r and Q_r are obtained directly from eq. (4.1) and thus their derivation does not require knowledge of lower-order "charges".

In compact form, the conservation laws for fixed n are the $(n+1)$ coefficients (equated to zero) of powers of λ in the matrix relation

$$[\partial_{\bar{y}}\quad \partial_{\bar{z}}](\mathbf{M}^{\lambda})^n\begin{bmatrix}J^{-1}J_y\\ J^{-1}J_z\end{bmatrix}=0\ . \tag{4.3}$$

As an example, let us examine the case $n=2$. Using eq. (3.1) with $\lambda_1=\lambda_2\equiv\lambda$, we find that

$$P_0=J^{-1}J_y\ ,\quad Q_0=J^{-1}J_z\ ;$$
$$P_1=L_1(J^{-1}J_y)+L_2(J^{-1}J_z)\ ,$$
$$Q_1=L_3(J^{-1}J_y)+L_4(J^{-1}J_z)\ ;$$
$$P_2=(L_1^2+L_2L_3)(J^{-1}J_y)+(L_1L_2+L_2L_4)(J^{-1}J_z),$$
$$Q_2=(L_3L_1+L_4L_3)(J^{-1}J_y)+(L_3L_2+L_4^2)(J^{-1}J_z).$$

As is easy to verify, this set of conservation laws is a proper subset of those obtained for $n\geqslant 3$.

Returning to the nine BTs of section 2, we see that construction of local conservation laws is now simply a matter of reading off the local operators $L_1, ..., L_4$ in each case. Thus, for example, the BT No. 7 with $L_1=L_4=z\partial_y-\bar{y}\partial_{\bar{z}}$, $L_2=0$, $L_3=1$, yields the following densities:

$$P_0=J^{-1}J_y\ ,\quad Q_0=J^{-1}J_z\ ;$$
$$P_1=(z\partial_y-\bar{y}\partial_{\bar{z}})(J^{-1}J_y)\ ,$$
$$Q_1=J^{-1}J_y+(z\partial_y-\bar{y}\partial_{\bar{z}})(J^{-1}J_z)\ ;$$
$$P_2=(z\partial_y-\bar{y}\partial_{\bar{z}})^2(J^{-1}J_y)\ ,$$
$$Q_2=2(z\partial_y-\bar{y}\partial_{\bar{z}})(J^{-1}J_y)+(z\partial_y-\bar{y}\partial_{\bar{z}})^2(J^{-1}J_z)\ ,$$

etc. The proof that these densities indeed satisfy eq. (4.2) whenever J satisfies eq. (2.1), is left to the reader.

If the gauge group is U(1), then eq. (2.1) becomes the Laplace/wave equation $\phi_{y\bar{y}}+\phi_{z\bar{z}}=0$, where $\phi=\ln J$. So by the method described above one gets conserved local densities for the wave equation which are linear in ϕ.

We note that, as mentioned earlier, the validity of the conservation laws (4.2) is independent of the integrability of the BTs, i.e., of whether the pair of equations (4.1) can actually be solved for K. Indeed, as can be seen from the matrix form (4.3) of the conservation law, what is required is that J satisfy the SDYM equation; it is not necessary that this solution conform to the integrability condition stated earlier needed to obtain J'. Thus it can be said that, for the purpose of constructing local conservation laws, the BTs themselves are more fundamental than their solutions.

5. Conclusion

The starting point of this paper was the set of nine one-parameter families of BTs for SDYM, derived in ref. [1]. We have shown that, from any one of these families, one may create an infinity of many-parameter families of BTs. This construction has the underlying structure of an infinite-dimensional commutative semigroup, where the group elements are families of BTs.

Thus, associated with the SDYM equation, there is an infinite-dimensional abelian "hidden" symmetry and an infinite collection of local conservation laws. Presumably these properties are further manifestations of the "complete integrability" of SDYM [11,12]. A rigorous proof of this statement, however, requires further investigation.

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Local Currents for the GL(N, C) Self-Dual Yang–Mills Equation

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Abstract. By using a simple Bäcklund-like transformation which linearizes the GL(N, C) self-dual Yang–Mills equation, an infinite number of local conservation laws for this equation are constructed. In the SL(N, C) case, the currents become trivial, which explains why these currents are not found in SU(N) gauge theory.

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As is well known, the self-dual Yang–Mills (SDYM) equation, when properly formulated, displays many of the typical characteristics of an 'integrable' system, such as parametric Bäcklund transformations [1–4], infinite number of nonlocal conservation laws [5, 6], linear system [7, 8, 6], Painlevé property [9, 10], etc. An infinite number of local conservation laws were recently constructed by these authors [11] by applying infinitesimal Bäcklund transformations [4] on SDYM (which is itself in the form of a local conservation law). However, the problem of finding *all* conservation laws for SDYM is far from being solved.

Indeed, the search for local currents for SDYM has been long and frustrating. It seems that the best we can do is write densities which are local in the Yang–Brihaye–Pohlmeyer function J [12, 13, 6]. But the latter is *nonlocal* in the potentials A_μ, and it is these potentials that are regarded as fundamental in the theory. Even in the J formulation there has been little progress in constructing local conservation laws.

Perhaps one of the reasons is that the search has been mostly confined to SU(N) [or, more generally, SL(N, C)] gauge theory, which requires unit determinant. This restriction ceases to exist in GL(N, C) theory. There the determinant itself becomes a field and, as we show in this Letter, can be used to produce new local conservation laws for the SDYM equation. In the limiting case of SL(N, C), the densities become zero and the conservation laws trivial, which explains why these objects are not found in SU(N) gauge theory.

The construction of the GL(N, C) conservation laws is based on a simple, known Bäcklund-like transformation that 'linearizes' the SDYM equation to the Laplace

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equation. The latter has an infinite number of variational 'Lie–Bäcklund' symmetries and associated conservation laws [14, 15]. The transformation then allows one to associate a local conservation law for SDYM with every conservation law of the Laplace equation.

Following Yang [12], Brihaye *et al.* [13], and Pohlmeyer [6], we write the SDYM equation as

$$\partial_{\bar{y}}(J^{-1}J_y) + \partial_{\bar{z}}(J^{-1}J_z) = 0, \tag{1}$$

(where $J_y = \partial_y J \equiv \partial J/\partial y$, etc.). The variables $y, z, \bar{y}, \bar{z}$, collectively denoted x^μ ($\mu = 1, 2, 3, 4$, respectively) are constructed from the coordinates of an underlying complexified Euclidean space in such a way that $\bar{y}$ and $\bar{z}$ become the complex conjugates of y and z, respectively, when the above space is real. The matrix function $J = J(x^\mu)$ will be assumed to have values in GL(N, C). For SU(N) gauge theory, J is required to be a positive Hermitian SL(N, C) matrix in real space. Here, J is also assumed to be a positive Hermitian matrix, but with a determinant which may vary with x^μ. (The simplest theory to which these latter conventions apply is U(1) electromagnetism.)

Taking the trace of Equation (1), and noting that

$$\mathrm{tr}(J^{-1}J_y) = \mathrm{tr}(\log J)_y = \partial_y[\mathrm{tr}(\log J)],$$

etc., we find the linear equation

$$\phi_{y\bar{y}} + \phi_{z\bar{z}} = 0, \tag{2}$$

where

$$\phi = \mathrm{tr}(\log J) = \log(\det J). \tag{3}$$

Clearly, ϕ becomes zero for SL(N, C)-valued J. Also, the hermiticity of J implies that ϕ is real in real space.

We note that Equation (3) is a sort of Bäcklund transformation that takes every solution J of the SDYM equation (1) into a solution of the linear equation (2). The latter equation is the four-dimensional Laplace equation when written in real coordinates. In our coordinates x^μ, Equation (2) is the Euler–Lagrange equation for the Lagrangian density

$$L = \tfrac{1}{2}(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}). \tag{4}$$

As is well known, every *variational* symmetry of the problem (but not necessarily every symmetry of the equation of motion) gives rise to a local conservation law. By 'variational symmetry' we mean an infinitesimal transformation

$$\delta\phi = \lambda F(x^\mu, \phi, \partial_\mu\phi, \partial_\mu\,\partial_\nu\phi, \ldots) \equiv \lambda F[\phi] \tag{5}$$

(where λ is an infinitesimal parameter) which shifts the Lagrangian by a divergence: $\delta L = \lambda\,\partial_\mu\Omega^\mu$. For every such symmetry, there exist local densities $P^\mu[\phi]$ ($\mu = 1, \ldots, 4$) such that

$$F[\phi](\phi_{y\bar{y}} + \phi_{z\bar{z}}) = \partial_\mu P^\mu[\phi]. \tag{6}$$

Clearly, Equation (6) becomes a local continuity equation for solutions ϕ of Equation (2).

The Lagrangian (4) has an infinite number of variational symmetries of the form (5) and an equal number of associated local conservation laws [14, 15]. Using transformation (3), we rewrite Equation (6) as

$$F[\phi(J)]\ \mathrm{tr}\{(J^{-1}J_y)_{\bar{y}} + (J^{-1}J_z)_{\bar{z}}\} = \partial_\mu P^\mu[\phi(J)], \tag{7}$$

for every conservation law of the Laplace equation (2). We thus obtain an infinite number of local currents for SDYM (local, of course, in J). It should be noted that these currents are different from those of [11] (in particular, the latter are matrix-valued and are not associated with the symmetries of a variational problem).

One may now ask the question: Can we associate the conservation laws (7) with symmetries of the SDYM equation? To this end we recall [16] that if J is a GL(N, C) solution of the SDYM equation, then

$$J' = (\det J)^{-1/N} J$$

is an SL(N, C) solution of this equation. [This is shown by using Equations (2) and (3), and by noticing that $\det J' = 1$.] We now see that *every* GL(N, C) solution can be reached from a corresponding SL(N, C) solution by means of a scalar transformation. Given an SL(N, C) solution J', the set of all GL(N, C) solutions J that can be reached from J' in this manner is given by

$$J = \mathrm{e}^{\phi/N} J' \tag{8}$$

where $\phi_{y\bar{y}} + \phi_{z\bar{z}} = 0$. [Then $\det J = \mathrm{e}^{\phi}$ and $\log(\det J)$ satisfies the Laplace equation, as required.]

Given a pair (ϕ, J'), we can construct an infinity of GL(N, C) solutions J, simply by employing the symmetries of the Laplace equation to find new solutions from the old solution ϕ. Thus, these symmetries are also symmetries of the GL(N, C) self-duality equation through Equation (8). We conclude that the infinite number of conservation laws (7) are associated with an infinite-dimensional symmetry of the SDYM equation.

As examples, let us write two of the conservation laws explicitly. [The reader may easily verify them by using the result that, since $\mathrm{tr}(J^{-1}J_u) = \phi_u$ (all u), one has the identity: $\mathrm{tr}(J^{-1}J_u)_v = \mathrm{tr}(J^{-1}J_v)_u$.]

1. The variational symmetry

$\delta\phi = \lambda(\phi_y + \phi_{\bar{y}})$ (i.e., $F[\phi] = \phi_y + \phi_{\bar{y}}$), corresponding, in real space, to translation of one of the Euclidean coordinates, yields the local conservation law

$$\begin{aligned}
&\partial_y\{-\mathrm{tr}(J^{-1}J_z)\ \mathrm{tr}(J^{-1}J_{\bar{z}}) + [\mathrm{tr}(J^{-1}J_{\bar{y}})]^2\} + \\
&\quad + \partial_z\{\mathrm{tr}(J^{-1}J_y + J^{-1}J_{\bar{y}})\ \mathrm{tr}(J^{-1}J_{\bar{z}})\} + \\
&\quad + \partial_{\bar{y}}\{[\mathrm{tr}(J^{-1}J_y)]^2 - \mathrm{tr}(J^{-1}J_z)\ \mathrm{tr}(J^{-1}J_{\bar{z}})\} + \\
&\quad + \partial_{\bar{z}}\{\mathrm{tr}(J^{-1}J_y + J^{-1}J_{\bar{y}})\ \mathrm{tr}(J^{-1}J_z)\} = 0.
\end{aligned}$$

2. The variational symmetry

$\delta\phi = \lambda(\phi_{yyy} + \phi_{\bar{y}\bar{y}\bar{y}})$ (with no geometrical analogue) yields the conservation law

$$\begin{aligned}
&\partial y\{\operatorname{tr}(J^{-1}J_y)_z \operatorname{tr}(J^{-1}J_y)_{\bar{z}} - [\operatorname{tr}(J^{-1}J_{\bar{y}})_{\bar{y}}]^2\} - \\
&\quad - \partial_z\{\operatorname{tr}(J^{-1}J_y)_y \operatorname{tr}(J^{-1}J_y)_{\bar{z}} + \operatorname{tr}(J^{-1}J_{\bar{y}})_{\bar{y}} \operatorname{tr}(J^{-1}J_{\bar{y}})_{\bar{z}}\} + \\
&\quad + \partial_{\bar{y}}\{-[\operatorname{tr}(J^{-1}J_y)_y]^2 + \operatorname{tr}(J^{-1}J_z)_{\bar{y}} \operatorname{tr}(J^{-1}J_{\bar{y}})_{\bar{z}}\} - \\
&\quad - \partial_{\bar{z}}\{\operatorname{tr}(J^{-1}J_y)_y \operatorname{tr}(J^{-1}J_y)_z + \operatorname{tr}(J^{-1}J_{\bar{y}})_{\bar{y}} \operatorname{tr}(J^{-1}J_z)_{\bar{y}}\} = 0.
\end{aligned}$$

(One similarly constructs local conservation laws for all variational symmetries of the form, chosen to keep $\delta\phi$ real,

$$\delta\phi = \lambda[(\partial_y)^{2k+1} + (\partial_{\bar{y}})^{2k+1}]\phi, \qquad \delta\phi = \lambda[(\partial_z)^{2k+1} + (\partial_{\bar{z}})^{2k+1}]\phi.)$$

Note that, in general, the conservation laws are trivially satisfied in the $SL(N, C)$ case.

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NONLOCAL CURRENTS FOR THE SELF-DUAL YANG–MILLS EQUATION

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New nonlocal conservation laws for the self-dual Yang–Mills equation are found by an inductive process. It appears that this process yields an infinite number of nonlocal currents, the densities of which depend on an increasing number of nonlocal charges.

As is well known, the self-dual Yang–Mills (SDYM) equation, when properly formulated, displays many of the typical characteristics of an "integrable" system, such as parametric Bäcklund transformations [1–4], infinite number of nonlocal conservation laws [5,6], linear system [6–8], Painlevé property [9,10], etc. An infinite number of local conservation laws were recently constructed [11] by applying infinitesimal Bäcklund transformations [4] on SDYM (which is itself in the form of a local conservation law). However, the problem of finding *all* conservation laws for SDYM is far from being solved.

This Letter reports the existence of additional nonlocal conserved currents for the SDYM equation. These currents are obtained by an inductive process which involves various integrability conditions and successive introduction of nonlocal "charges". To be more specific, one starts with the SDYM equation and finds a simple non-auto-Bäcklund transformation that relates SDYM with a nonlocal conservation law depending on a nonlocal charge. Then another Bäcklund transformation is introduced which relates the aforementioned conservation law with a new one depending on an additional charge, and so forth. Although it appears that the above-described process can be continued indefinitely, no recursion relation seems to exist that allows construction of current densities in terms of lower-order charges. This makes higher-order densities very difficult to obtain.

Following Yang [12], Brihaye et al. [13], and Pohlmeyer [6], we write the SDYM equation as

$$F(J) \equiv \partial_{\bar{y}}(J^{-1}J_y) + \partial_{\bar{z}}(J^{-1}J_z) = 0 \tag{1}$$

(where $J_y = \partial_y J = \partial J/\partial y$, etc.). The variables y, z, $\bar{y}$, $\bar{z}$ are constructed from the coordinates of an underlying complexified Euclidean space in such a way that $\bar{y}$ and $\bar{z}$ become the complex conjugates of y and z, respectively, when the above space is real. The matrix function J is generally assumed to have values in $GL(N, C)$. For $SU(N)$ gauge theory, J is required to be a Hermitian $SL(N, C)$ matrix in real space.

The SDYM equation $F(J)=0$ is in the form of a local conservation law for J. A new conservation law can be found by employing the simple Bäcklund transformation

$$J^{-1}J_y = X_{\bar{z}}, \quad J^{-1}J_z = -X_{\bar{y}}. \tag{2}$$

The integrability condition $(X_{\bar{y}})_{\bar{z}} = (X_{\bar{z}})_{\bar{y}}$ of the system (2) yields the SDYM equation (1). The integrability condition $(J_y)_z = (J_z)_y$, or equivalently,

$$\partial_y(J^{-1}J_z) - \partial_z(J^{-1}J_y) + [J^{-1}J_y, J^{-1}J_z] = 0,$$

yields a nonlinear equation for X:

$$X_{y\bar{y}} + X_{z\bar{z}} - [X_{\bar{y}}, X_{\bar{z}}] = 0. \tag{3}$$

With the observation that

$$[X_{\bar{y}}, X_{\bar{z}}] = \tfrac{1}{2}(\partial_{\bar{y}}[X, X_{\bar{z}}] - \partial_{\bar{z}}[X, X_{\bar{y}}]),$$

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eq. (3) is written in the form of a continuity equation:

$$\partial_{\bar{y}}(X_y - \tfrac{1}{2}[X, X_{\bar{z}}]) + \partial_{\bar{z}}(X_z + \tfrac{1}{2}[X, X_{\bar{y}}]) = 0. \quad (4)$$

Substituting for $X_{\bar{y}}$ and $X_{\bar{z}}$ from eqs. (2), we finally get

$$\partial_{\bar{y}}(X_y + \tfrac{1}{2}[J^{-1}J_y, X]) + \partial_{\bar{z}}(X_z + \tfrac{1}{2}[J^{-1}J_z, X]) = 0. \quad (5)$$

By expanding the left-hand side of eq. (5), and using eqs. (2), we rewrite eq. (5) as

$$\tfrac{1}{2}[F(J), X] = 0.$$

We thus conclude that eq. (5) is a nontrivial [14], nonlocal (due to X) conservation law which is satisfied on all SDYM solutions J. The densities of the conserved current depend explicitly on the nonlocal "charge" X.

Let us compare eq. (5) with the corresponding nonlocal conservation law of Prasad et al. [5] and Pohlmeyer [6]:

$$\partial_{\bar{y}}(X_y + J^{-1}J_y X) + \partial_{\bar{z}}(X_z + J^{-1}J_z X) = 0. \quad (6)$$

Subtracting this from eq. (5), we get

$$\partial_{\bar{y}}\{J^{-1}J_y, X\} + \partial_{\bar{z}}\{J^{-1}J_z, X\} = 0, \quad (7)$$

where the curly brackets denote anticommutators. As can be easily verified, eq. (7) is a nontrivial conservation law for SDYM. Thus we conclude that the conservation law (5) is not trivially related (i.e., equivalent [14]) to the familiar conservation law (6).

Let us return to the continuity equation (4). We seek a Bäcklund transformation, one of the integrability conditions of which is eq. (4) while another integrability condition yields a higher-order continuity equation. The obvious choice is

$$X_y - \tfrac{1}{2}[X, X_{\bar{z}}] = \Phi_{\bar{z}}, \quad X_z + \tfrac{1}{2}[X, X_{\bar{y}}] = -\Phi_{\bar{y}}. \quad (8)$$

The consistency condition $(X_y)_z = (X_z)_y$ yields, after some calculation,

$$\Phi_{y\bar{y}} + \Phi_{z\bar{z}} + \tfrac{1}{2}([\Phi_{\bar{z}}, X_{\bar{y}}] - [\Phi_{\bar{y}}, X_{\bar{z}}]) + \tfrac{1}{4}[X, [X_{\bar{y}}, X_{\bar{z}}]] = 0. \quad (9)$$

With the observation that

$$[\Phi_{\bar{z}}, X_{\bar{y}}] - [\Phi_{\bar{y}}, X_{\bar{z}}] = \partial_{\bar{z}}[\Phi, X_{\bar{y}}] - \partial_{\bar{y}}[\Phi, X_{\bar{z}}],$$

$$[X, [X_{\bar{y}}, X_{\bar{z}}]] = \tfrac{1}{3}(\partial_{\bar{y}}[X, [X, X_{\bar{z}}]] - \partial_{\bar{z}}[X, [X, X_{\bar{y}}]]),$$

eq. (9) takes the form of a continuity equation:

$$\partial_{\bar{y}}(\Phi_y - \tfrac{1}{2}[\Phi, X_{\bar{z}}] + \tfrac{1}{12}[X, [X, X_{\bar{z}}]]) + \partial_{\bar{z}}(\Phi_z + \tfrac{1}{2}[\Phi, X_{\bar{y}}] - \tfrac{1}{12}[X, [X, X_{\bar{y}}]]) = 0. \quad (10)$$

Substituting for $X_{\bar{y}}$ and $X_{\bar{z}}$ from eqs. (2), we finally have

$$\partial_{\bar{y}}(\Phi_y + \tfrac{1}{2}[J^{-1}J_y, \Phi] + \tfrac{1}{12}[X, [X, J^{-1}J_y]]) + \partial_{\bar{z}}(\Phi_z + \tfrac{1}{2}[J^{-1}J_z, \Phi] + \tfrac{1}{12}[X, [X, J^{-1}J_z]]) = 0. \quad (11)$$

By expanding eq. (11), and using eqs. (2) and (8), one finds

$$\tfrac{1}{2}[F(J), \Phi] + \tfrac{1}{12}[X, [X, F(J)]] = 0,$$

which confirms that eq. (11) is a nonlocal conservation law for SDYM.

To find the next conservation law, we return to the continuity equation (10) and notice that this expression is a consistency condition for the Bäcklund transformation

$$\Phi_y - \tfrac{1}{2}[\Phi, X_{\bar{z}}] + \tfrac{1}{12}[X, [X, X_{\bar{z}}]] = P_{\bar{z}},$$
$$\Phi_z + \tfrac{1}{2}[\Phi, X_{\bar{y}}] - \tfrac{1}{12}[X, [X, X_{\bar{y}}]] = -P_{\bar{y}}. \quad (12)$$

We then apply the other integrability condition, $(\Phi_y)_z = (\Phi_z)_y$, and use eqs. (8) to eliminate X_y and X_z. After a very lengthy calculation, the result is rewritten in the form of a continuity equation which, with the aid of eqs. (2), takes the final form

$$\begin{aligned}&\partial_{\bar{y}}(P_y + \tfrac{1}{2}[J^{-1}J_y, P] + \tfrac{1}{12}[\Phi, [X, J^{-1}J_y]] \\ &\quad + \tfrac{1}{12}[X, [\Phi, J^{-1}J_y]] + \tfrac{1}{24}[X, [X, [X, J^{-1}J_y]]] \\ &\quad - \tfrac{1}{12}X^3J^{-1}J_y + \tfrac{1}{12}X^2J^{-1}J_yX - \tfrac{1}{6}XJ^{-1}J_yX^2) \\ &\quad + \partial_{\bar{z}}(P_z + \tfrac{1}{2}[J^{-1}J_z, P] + \tfrac{1}{12}[\Phi, [X, J^{-1}J_z]] \\ &\quad + \tfrac{1}{12}[X, [\Phi, J^{-1}J_z]] + \tfrac{1}{24}[X, [X, [X, J^{-1}J_z]]] \\ &\quad - \tfrac{1}{12}X^3J^{-1}J_z + \tfrac{1}{12}X^2J^{-1}J_zX - \tfrac{1}{6}XJ^{-1}J_zX^2) = 0. \end{aligned} \quad (13)$$

By expanding eq. (13), and using eqs. (2), (8), and (12), the reader may verify that eq. (13) is a nonlocal conservation law for SDYM.

We have thus established a process for generating nonlocal conservation laws for SDYM. Presumably, an infinite number of currents can be obtained in this fashion, although a rigorous proof of this statement requires further investigation. Unfortunately, the construction of higher-order conservation laws is an increasingly hard task since it becomes excessively difficult to express the ensuing relations in the form of continuity equations. On the other hand, there seems to exist no recursion relation that allows expression of current densities in terms of lower-order charges. In spite of these difficulties, the existence of these conservation laws is a welcome addition to the long list of "integrability" characteristics of the SDYM equation, and it may suggest that there are additional "hidden" symmetries of SDYM besides those already known [15].

Finally, it is of interest to compare our nonlocal conservation laws with those of Prasad et al. [5]. Let us first summarize the results of ref. [5].

The SDYM equation (1) is in the form of a continuity equation,

$$\partial_{\bar{y}}A^{(1)}+\partial_{\bar{z}}B^{(1)}=0, \tag{14}$$

where $A^{(1)}=J^{-1}J_y$, $B^{(1)}=J^{-1}J_z$. Eq. (14) guarantees that there exists a complex matrix function $X^{(1)}$ such that

$$A^{(1)}=\partial_{\bar{z}}X^{(1)},\quad B^{(1)}=-\partial_{\bar{y}}X^{(1)}. \tag{15}$$

Let us assume we have constructed the nth continuity equation

$$\partial_{\bar{y}}A^{(n)}+\partial_{\bar{z}}B^{(n)}=0. \tag{16}$$

This implies

$$A^{(n)}=\partial_{\bar{z}}X^{(n)},\quad B^{(n)}=-\partial_{\bar{y}}X^{(n)}, \tag{17}$$

for some matrix function $X^{(n)}$. The $(n+1)$th continuity equation is then defined as

$$\partial_{\bar{y}}A^{(n+1)}+\partial_{\bar{z}}B^{(n+1)}=0, \tag{18a}$$

$$A^{(n+1)}=(\partial_y+J^{-1}J_y)X^{(n)}, \tag{18b}$$

$$B^{(n+1)}=(\partial_z+J^{-1}J_z)X^{(n)}. \tag{18c}$$

Thus, for example, the first nonlocal conservation law (corresponding to $n=1$) is

$$\partial_{\bar{y}}(X_y^{(1)}+J^{-1}J_yX^{(1)}) +\partial_{\bar{z}}(X_z^{(1)}+J^{-1}J_zX^{(1)})=0. \tag{19}$$

By comparing eqs. (2) and (15) we observe that $X^{(1)}=X$, i.e., our first nonlocal charge X coincides with $X^{(1)}$. (Note also that eqs. (6) and (19) are identical.)

By expressing $A^{(n)}$ and $B^{(n)}$ in terms of $X^{(n-1)}$, eqs. (17) become a system of equations which play a role analogous to that of our Bäcklund transformations. The only difference is that now these transformations are essentially the same for all steps of the recursive process (that is, for all values of the index n). Thus the Prasad et al. conservation laws can be evaluated via a recursion relation for all values of n, which is not the case with our currents. Note also that the latter are much more complicated than the former since their densities depend on an increasing number of charges rather than on one charge at a time.

We would like to emphasize that relations like eq. (15) or, more generally, eq. (17), do not imply that conservation laws such as (14) or (16) are trivial. Indeed, the existence of a charge $X^{(n)}$, satisfying eq. (17), is automatically guaranteed by the conservation law (16).

As long as $X^{(n)}$ is *nonlocally* related to J, this conservation law is nontrivial. (Of course, if $X^{(n)}$ were locally dependent on J and its partial derivatives, then the densities $A^{(n)}$ and $B^{(n)}$ would be perfect derivatives of a local function and the charge $X^{(n)}$ would become trivial over the entire Euclidean space.) Therefore, the various Bäcklund transformations that we have introduced do not have the effect of trivializing the corresponding nonlocal conservation laws.

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POTENTIAL SYMMETRIES FOR SELF-DUAL GAUGE FIELDS

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New symmetries of the self-dual Yang–Mills equation (SDYM) are reported. They are associated with the infinite set of symmetries of a closely related equation which we call the potential SDYM equation. These symmetries lead, in a remarkably simple way, to the construction of infinite collections of conserved Noether-like currents for the SDYM.

1. Introduction

The concept of *potential symmetries* is a relatively new one in the theory of partial differential equations (PDEs) (see, for example, ref. [1] and the extensive references therein). These symmetries are realized as nonlocal transformations of the dependent variable of a PDE which depend explicitly on the potential of a conservation law associated with this PDE. Often, the PDE itself is in conservation-law form.

This paper describes a method for obtaining potential symmetries for the self-dual Yang–Mills equation (SDYM). Previous studies of the SDYM have revealed a finite-dimensional group of point symmetries [2]; certain of which can be written in the form of infinitesimal Lie–Bäcklund transformations [3], together with an infinite-dimensional "hidden" symmetry of nonlocal transformations [4–6]. The latter transformations are excellent examples of potential symmetries. However, as we show in this paper, this set is far from being exhaustive. In fact, there is a whole collection of infinite sets of potential symmetries of the SDYM, each set associated with a symmetry of what we call the potential SDYM equation (PSDYM).

In addition to giving new symmetries, our method naturally leads to the construction of infinite sets of (generally nonlocal) conserved currents for SDYM. These currents are complementary to those found previously by this author and others [6–11]. The fact that the currents are associated with symmetries suggests the existence of an underlying Noether-like structure for this problem.

The paper is organized as follows.

In section 2, the PSDYM equation is derived and its symmetry characteristics are shown to satisfy a certain PDE. A fundamental theorem is then proven which states that, from every symmetry of the PSDYM, one may construct (in a certain way) a symmetry of the SDYM, and vice versa.

In section 3, a Bäcklund transformation (BT) is presented which generates infinite sets of symmetries for the PSDYM, and thus also for the SDYM.

In section 4, the BT is used to construct infinite collections of conserved currents for the SDYM.

Finally, section 5 contains examples of the use of the BT to produce new symmetries, as well as examples of conservation laws associated with certain of these symmetries.

2. Potential SDYM equation (PSDYM)

Following Yang [12], Brihaye et al. [13], and Pohlmeyer [8], we write the SDYM equation as

$$F(J) \equiv \partial_{\bar{y}}(J^{-1}J_y) + \partial_{\bar{z}}(J^{-1}J_z) = 0 , \qquad (2.1)$$

(where $J_y = \partial_y J = \partial J/\partial y$, etc.). The variables y, z, $\bar{y}$, $\bar{z}$ are constructed from the coordinates of an under-

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lying complexified Euclidean space in such a way that $\bar{y}$ and $\bar{z}$ become the complex conjugates of y and z, respectively, when the above space is real. The matrix function J is generally assumed to have values in GL(N, C). For SU(N) gauge theory, J is required to be a Hermitian SL(N, C) matrix in real space.

We note that the SDYM equation is a consistency condition for the non-auto-Bäcklund transformation

$$J^{-1}J_y = X_{\bar{z}}, \quad J^{-1}J_z = -X_{\bar{y}}, \tag{2.2}$$

where X is a matrix function. The other consistency condition is found by application of the identity

$$\partial_u(J^{-1}J_v) - \partial_v(J^{-1}J_u) + [J^{-1}J_u, J^{-1}J_v] = 0, \tag{2.3}$$

with $u=y$, $v=z$. This yields the following nonlinear PDE for X,

$$G(X) \equiv X_{y\bar{y}} + X_{z\bar{z}} - [X_{\bar{y}}, X_{\bar{z}}] = 0, \tag{2.4}$$

which we call the *potential SDYM equation* (PSDYM). (The reason for this name is that, according to eqs. (2.2), X is a potential to the conservation law (2.1), which law is precisely the SDYM equation.)

Let

$$X' = X + \alpha\Phi \quad \text{or} \quad \delta X = \alpha\Phi, \tag{2.5}$$

be an infinitesimal transformation which preserves the PSDYM (here α is an infinitesimal parameter and Φ is a matrix function). The condition that $G(X')=0$ when $G(X)=0$ implies that, *for a given solution X of $G(X)=0$*, Φ must satisfy the following PDE,

$$H(\Phi) \equiv \Phi_{y\bar{y}} + \Phi_{z\bar{z}} + [X_{\bar{z}}, \Phi_{\bar{y}}] - [X_{\bar{y}}, \Phi_{\bar{z}}] = 0, \tag{2.6}$$

or, using eqs. (2.2),

$$H(\Phi) \equiv \Phi_{y\bar{y}} + \Phi_{z\bar{z}} + [J^{-1}J_y, \Phi_{\bar{y}}] + [J^{-1}J_z, \Phi_{\bar{z}}] = 0. \tag{2.7}$$

The following theorem relates the symmetries of the PSDYM with those of the SDYM.

Theorem 2.1. Let $\delta X = \alpha\Phi$ be a symmetry of $G(X)=0$. Then, $\delta J = \beta J\Phi$ is a symmetry of $F(J)=0$. Conversely, if $\delta J = \beta Q$ is a symmetry of $F(J)=0$, then $\delta X = \alpha J^{-1}Q$ is a symmetry of $G(X)=0$.

Proof. (a) Let $\delta X = \alpha\Phi$ be a symmetry of $G(X)=0$. Consider the transformation $\delta J = \beta J\Phi$. Then

$$\delta(J^{-1}J_y) = \beta(\Phi_y + [J^{-1}J_y, \Phi]),$$

$$\delta(J^{-1}J_z) = \beta(\Phi_z + [J^{-1}J_z, \Phi]),$$

and, differentiating with respect to $\bar{y}$ and $\bar{z}$ and adding,

$$\delta F(J) = \beta[F(J), \Phi] + \beta H(\Phi),$$

which vanishes in view of eqs. (2.1) and (2.7).

(b) Let $\delta J = \beta Q$ be a symmetry of $F(J)=0$. We put: $Q = J(J^{-1}Q) \equiv J\Phi$. Then

$$\Delta F(J) = \beta[F(J), \Phi] + \beta H(\Phi) = 0.$$

Given that $F(J)=0$, it follows that $H(\Phi)=0$, according to which, $\delta X = \alpha\Phi = \alpha J^{-1}Q$ is a symmetry of $G(X)=0$.

Corollary 2.1. There is a one-to-one correspondence between the infinitesimal symmetries of the SDYM and those of the PSDYM.

It should be noted that [1,14] *all* symmetries of a PDE can be expressed as infinitesimal transformations of the dependent variable alone. In more technical terms, every symmetry of a PDE is equivalent to a "vertical" or "evolutionary" symmetry. Thus there is no loss of generality if all infinitesimal symmetries are expressed in the form (2.5).

3. Bäcklund transformation for symmetries

To find symmetries of the PSDYM (and thus also of the SDYM, according to theorem 2.1) one must integrate the second-order PDEs (2.6) or (2.7) for Φ, keeping in mind that the functions X and J, appearing in the above PDEs, represent the original, untransformed solutions of the PSDYM and the SDYM, respectively. We now present a parametric Bäcklund transformation which generates solutions of (2.6) or (2.7) *for given functions X or J.*

Theorem 3.1. The pair of first-order PDEs

$$\lambda\Phi'_{\bar{z}} = \Phi_y + [X_{\bar{z}}, \Phi], \quad \lambda\Phi'_{\bar{y}} = -\Phi_z + [X_{\bar{y}}, \Phi] \tag{3.1}$$

(where λ is an arbitrary (finite) complex parameter) is a strong auto-Bäcklund transformation (BT) for the PDE: $H(\Phi)=0$, for any given PSDYM solution X.

Proof. (a) The integrability condition $\Phi'_{\bar{z}\bar{y}}=\Phi'_{\bar{y}\bar{z}}$ yields $H(\Phi)=0$.

(b) The integrability condition $\Phi_{yz}=\Phi_{zy}$ yields

$$\lambda H(\Phi')+[\Phi, G(X)]=0\,,$$

which implies $H(\Phi')=0$.

By using eqs. (2.2), the BT (3.1) is rewritten as

$$\lambda\Phi'_{\bar{z}}=\Phi_y+[J^{-1}J_y, \Phi]\,,$$
$$-\lambda\Phi'_{\bar{y}}=\Phi_z+[J^{-1}J_z, \Phi]\,. \qquad (3.2)$$

Corollary 3.1. Let Φ and Φ' be a pair of functions which satisfy the BT (3.2) for a given SDYM solution J. Then the infinitesimal transformations $\delta J=\alpha J\Phi$ and $\delta J=\alpha J\Phi'$ independently preserve the SDYM equation.

According to the above corollary, the expressions $J'=J+\alpha J\Phi$ and $J''=J+\alpha J\Phi'$ both are SDYM solutions. Thus, by expressing Φ and Φ' in terms of J, J' and J, J'', respectively, theorem 3.1 can be restated as follows.

Theorem 3.2. Consider the pair of PDEs

$$\lambda(J^{-1}J'')_{\bar{z}}=(J^{-1}J')_y+[J^{-1}J_y, J^{-1}J']\,,$$
$$-\lambda(J^{-1}J'')_{\bar{y}}=(J^{-1}J')_z+[J^{-1}J_z, J^{-1}J']\,, \qquad (3.3)$$

where J is an SDYM solution, and where the matrix functions J' and J'' differ *infinitesimally* from J. Then, both J' and J'' are solutions of SDYM.

Note, in particular, that the infinitesimal BT (3.3) is a *strong* BT, in the sense that the SDYM properties are satisfied independently for J' and J'', for any given SDYM solution J. Thus the old question, whether the SDYM possesses a strong BT, has been answered in the affirmative.

4. Conservation laws

Given any symmetry of the PSDYM, an infinite number of symmetries can be constructed by repeated application of the BT (3.1) or, equivalently, the BT (3.2). Thus, let $\Phi^{(0)}$ be a solution of $H(\Phi)=0$. An infinite sequence of solutions $\Phi^{(n)}$ ($n=0, 1, 2, ...$) can be formed recursively by using the BTs,

$$\lambda\Phi^{(n+1)}_{\bar{z}}=\Phi^{(n)}_y+[J^{-1}J_y, \Phi^{(n)}]\,,$$
$$-\lambda\Phi^{(n+1)}_{\bar{y}}=\Phi^{(n)}_z+[J^{-1}J_z, \Phi^{(n)}]\,, \qquad (4.1)$$

for any given SDYM solution J.

As an example, consider the obvious symmetry $\delta X=\alpha T^{(0)}$, where $T^{(0)}$ is a constant matrix and α is an infinitesimal parameter. Putting $\Phi^{(0)}=T^{(0)}$, $\lambda=1$, and integrating the BTs (4.1) recursively, we obtain an infinite sequence of matrix functions $T^{(n)}$. Application of theorem 2.1 then yields an infinite number of symmetries of the SDYM, of the form $\delta^{(n)}J=\beta JT^{(n)}$ (where β is an infinitesimal parameter). Putting $\beta T^{(n)}\equiv-\Lambda^{(n)}$, where $\Lambda^{(n)}$ is an infinitesimal matrix function, we finally obtain

$$\delta^{(n)}J=-J\Lambda^{(n)}\,,\quad n=0, 1, 2, ...\,. \qquad (4.2)$$

We have thus recovered the "hidden" (Kac–Moody) symmetry of the SDYM [4–6] as a special case of our symmetry-generating process.

Returning to the general BT (4.1), we note that its integrability with respect to $\Phi^{(n+1)}$ for all $n=0$, 1, 2, ..., is equivalent to an infinite set of continuity equations:

$$\partial_{\bar{y}}(\Phi^{(n)}_y+[J^{-1}J_y, \Phi^{(n)}])$$
$$+\partial_{\bar{z}}(\Phi^{(n)}_z+[J^{-1}J_z, \Phi^{(n)}])=0\,. \qquad (4.3)$$

Thus we obtain an infinite sequence of conservation laws for the SDYM, corresponding to the infinite sequence of symmetries $\delta^{(n)}X=\alpha\Phi^{(n)}$ of the PSDYM. We comment that the result (4.3) is valid for *all* symmetry characteristics Φ of the PSDYM, not just those obtained from each other via the BT (4.1).

In characteristic form [14], the conservation laws (4.3) are written

$$[F(J), \Phi^{(n)}]=0 \qquad (4.4)$$

(this is verified by expanding the left-hand side of eq. (4.3) and using eq. (2.7)). We note that $\Phi^{(n)}$ is a sort of characteristic function for the conservation law (4.3), which function is also proportional to a characteristic of a symmetry of the SDYM. It may thus be conjectured that, associated with the currents (4.3), there is an underlying Noether-like structure. This last statement will be further explored in future articles.

5. Examples of potential symmetries

In this section we illustrate the use of the BTs (3.1) or (3.2), in combination with theorem 2.1, to generate new symmetries for the SDYM. Our method is outlined as follows. We start with a known symmetry $\delta J=\beta Q$ of the SDYM. Then, $\delta X=\alpha\Phi$, with $\Phi=J^{-1}Q$, is a symmetry of the PSDYM. Application of the BT yields a new function Φ', which in turn implies a new symmetry $\delta J=\beta J\Phi'$ for the SDYM. This symmetry is a genuine potential symmetry if Φ' depends explicitly on nonlocal variables such as X, X_y, X_z, etc.

(1) Consider the symmetry $\delta J=\beta J$ which represents a global phase change of J. Clearly, $\Phi=1$, where 1 is the unit matrix. Application of the BT (3.1) yields $\Phi'=M(y,z)$, where the matrix function M is arbitrary. The corresponding SDYM symmetry is a familiar one [2]: $\delta J=\beta JM(y,z)$.

Following the prescription (4.1), we integrate the BT (3.1) once more to find a new function Φ'' and a new symmetry $\delta J=\beta J\Phi''$. Explicitly,

$$\delta J=\frac{\beta}{\lambda}J\{\bar{z}M_y-\bar{y}M_z+[X,M(y,z)]+N(y,z)\},$$

where the matrix function N is arbitrary. The above symmetry is a genuine potential symmetry for the SDYM.

(2) Let $\delta J=\beta J_y$, which corresponds to a translation of the y coordinate [3]. Then, $\Phi=J^{-1}J_y=X_{\bar{z}}$. Application of the BT (3.1) yields (ignoring an arbitrary function of y and z) $\lambda\Phi'=X_y$, and therefore

$$\delta J=\frac{\beta}{\lambda}JX_y\equiv\alpha JX_y .$$

The corresponding conservation law is

$$\partial_{\bar{y}}(X_{yy}+[J^{-1}J_y,X_y])+\partial_{\bar{z}}(X_{yz}+[J^{-1}J_z,X_y])=0 .$$

(3) Let $\delta J=\beta(yJ_y+zJ_z)$, which corresponds to a scale change of y and z [3]. Then

$$\Phi=yJ^{-1}J_y+zJ^{-1}J_z=yX_{\bar{z}}-zX_{\bar{y}} .$$

Application of the BT (3.1) yields (ignoring, again, an arbitrary function of y and z)

$$\lambda\Phi'=X+yX_y+zX_z ,$$

and therefore, putting $\alpha=\beta/\lambda$,

$$\delta J=\alpha J(X+yX_y+zX_z) .$$

The corresponding conservation law is of the form $A_{\bar{y}}+B_{\bar{z}}=0$, where the densities A and B are given by

$$A=2X_y+yX_{yy}+zX_{yz}+[J^{-1}J_y,X+yX_y+zX_z] ,$$

$$B=2X_z+yX_{yz}+zX_{zz}+[J^{-1}J_z,X+yX_y+zX_z] .$$

(4) Let $\delta J=\beta(\bar{y}J_{\bar{y}}+\bar{z}J_{\bar{z}})$, which corresponds to a scale change of $\bar{y}$ and $\bar{z}$ [3]. Then

$$\Phi=\bar{y}J^{-1}J_{\bar{y}}+\bar{z}J^{-1}J_{\bar{z}} .$$

With the aid of identity (2.3) and eqs. (2.2), the BT (3.2) is integrated to give

$$-\lambda\Phi'=X+\bar{y}J^{-1}J_z-\bar{z}J^{-1}J_y$$

and therefore, putting $\alpha=-\beta/\lambda$,

$$\delta J=\alpha(JX+\bar{y}J_z-\bar{z}J_y) .$$

The corresponding conservation law is $A_{\bar{y}}+B_{\bar{z}}=0$, where

$$A=X_y+[J^{-1}J_y,X]+(\bar{y}\partial_z-\bar{z}\partial_y)J^{-1}J_y ,$$

$$B=X_z+[J^{-1}J_z,X]+(\bar{y}\partial_z-\bar{z}\partial_y)J^{-1}J_z .$$

(For the construction of these densities, the identity (2.3) has been used.)

6. Conclusion

The results of this paper indicate that the symmetry group of the SDYM can be enhanced by including the so-called potential symmetries [1]. We have defined the latter symmetries to be infinitesimal transformations of the SDYM function J which depend explicitly on the nonlocal potential of an associated conservation law.

To find such symmetries, we have employed a non-auto-Bäcklund transformation to transform the SDYM into the PSDYM (a PDE for the potential X of the SDYM itself) and then shown that every symmetry of the PSDYM corresponds to a symmetry of the SDYM, and vice versa. By using an appropriate

Bäcklund transformation, we have constructed (in principle) an infinite number of symmetries for the PSDYM which, in general, are potential symmetries of the SDYM. (The familiar Kac–Moody symmetry of the SDYM was seen to be a particular subsymmetry of this larger invariance set.) Finally, our method yielded an infinite number of Noether-like conserved matrix currents for the SDYM.

The reader may wonder whether these potential symmetries could have been predicted by using systematic techniques for finding symmetries of PDEs. It has been argued [15] that the best way to deal with symmetries of matrix-structured PDEs is to express the PDEs as matrix-valued differential forms. Much work on the geometrical derivation of potential symmetries for scalar PDEs was previously done by Kersten [16]. One thus needs to extend Kersten's methods to PDEs associated with matrix-valued exterior differential forms. This interesting geometrical problem and its application to the SDYM will be studied in future papers.

It is remarkable that the presence of an infinite number of symmetries is associated with the existence of a strong Bäcklund transformation, eq. (3.3). This is a situation frequently encountered in connection with integrable nonlinear systems. There is, however, a peculiarity in the present case, namely, the Bäcklund transformation is an *infinitesimal* one, producing solutions infinitesimally close to a given solution. In this respect, the BT (3.3) may be regarded as a sort of "recursion operator" [1,14] which generates infinite collections of symmetries for the SDYM.

In conclusion, the appearance of a larger set of symmetries and corresponding conservation laws significantly enhances the (already long) list of integrability characteristics of the SDYM, and promises new directions in the search for exact solutions. Two questions have been left open and require further investigation; namely, (a) how many *independent* infinite-parameter sets of symmetries there are, and (b) what is the underlying Lie algebraic structure associated with these symmetries. A prerequisite for answering these questions is a careful examination of the generalized (Lie–Bäcklund) symmetry group of the PSDYM and an identification of those symmetries which are not related to each other via a Bäcklund transformation. The detailed calculations providing these results will appear elsewhere.

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Recursion operator and current algebras for the potential SL(N, C) self-dual Yang–Mills equation

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The Lie algebraic structure of the recently reported potential symmetries for the self-dual Yang–Mills equation (SDYM) is explored. This structure contains both Kac–Moody and Virasoro algebras, formed by application of a recursion operator to the group of point transformations of the potential SDYM equation.

In a recent paper [1] we reported that the self-dual Yang–Mills equation (SDYM) possesses an infinite number of potential symmetries [2]. These symmetries are expressed as nonlocal transformations which depend on the potential of the SDYM equation. This potential satifies an equation of its own which we called the potential SDYM equation (PSDYM). We showed that there is a one-to-one correspondence between the symmetries of the SDYM and those of the PSDYM. In particular, the potential symmetries of the SDYM correspond to those symmetries of the PSDYM which depend nonlocally on the SDYM solution. We conjectured that these symmetries are of variational nature since each of them is associated with a Noether-like current.

Two important questions were raised in the conclusion of ref. [1]: (a) what is the complete set of potential symmetries of the SDYM; and (b) what is the Lie algebraic structure of these symmetries. Both questions are being answered in the present paper. In particular, the invariance group of the PSDYM is shown to have a rich Lie algebraic structure containing infinite-dimensional subalgebras of Kac–Moody and Virasoro types (both of which are examples of so-called current algebras [3]). These algebraic structures are known to be of considerable importance in quantum physics and string field theories, as well as in the theory of exactly solvable nonlinear equations [3–5].

A fundamental element in our analysis is a linear operator which, applied recursively, produces an infinite hierarchy of symmetries from any given one. This object is called a *recursion operator* [2]. The existence of such an operator for the PSDYM was noted in ref. [1], but a formal definition and a list of basis properties are given here for the first time. The nonlocal character of this operator is responsible for the increasingly nonlocal nature of the potential symmetries and the corresponding conservation laws [1].

Our main results are based on two theorems stating the conditions under which a point symmetry of the PSDYM generates a current algebra of Kac–Moody or Virasoro type. The proofs of these theorems are long and cannot be accommodated in full in a short communication. These proofs will be merely outlined here and will be presented in detail in a future, more extensive article.

We write the SDYM equation in the form [6–8]

$$D_{\bar{y}}(J^{-1}J_y)+D_{\bar{z}}(J^{-1}J_z)=0\,. \qquad (1)$$

The variables y, z, $\bar{y}$, $\bar{z}$, collectively denoted x^μ ($\mu=1$, 2, 3, 4, respectively), are constructed from the coordinates of an underlying complexified Euclidean space in such a way that $\bar{y}$ and $\bar{z}$ become the complex conjugates of y and z, respectively, when the above

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space is real. Total derivative operators (in the jet-space sense) with respect to these variables are denoted by D_y, etc. We also adopt the standard notation: $D_yF \equiv F_y$, etc., for any function F on the jet space. The variable J is, in general, an N-dimensional complex, nonsingular matrix. For real SU(N) gauge theory, J is required to be a Hermitian SL(N, C) matrix in real space.

Eq. (1) may be rewritten in potential form with the aid of the non-auto-Bäcklund transformation

$$J^{-1}J_y = X_{\bar{z}}, \quad J^{-1}J_z = -X_{\bar{y}} . \tag{2}$$

Integrability with respect to X implies eq. (1), while the consistency condition for J yields the nonlinear equation

$$G[X] \equiv X_{y\bar{y}} + X_{z\bar{z}} - [X_{\bar{y}}, X_{\bar{z}}] = 0 . \tag{3}$$

Since X is a potential for eq. (1), we call eq. (3) the potential SDYM equation (PSDYM). From eqs. (2) it follows that the condition $\det J = 1$ is satisfied if X is chosen to be traceless. Hence, SL(N, C) SDYM solutions correspond to sl(N, C) PSDYM solutions.

Let $\{V_r\}$ be a set of vector fields generating symmetries of the sl(N, C) PSDYM. It is possible [2] to express all symmetries by "vertical" vector fields, i.e., vectors with vanishing projections to the space of independent variables x^μ. The V_r may be viewed as linear operators acting on functionals $F[X]$ (which may be local or nonlocal in X). The Lie derivative of a functional $F[X]$, with respect to a vector V_r, is defined by

$$\Delta^{(r)}F[X] \equiv V_r\{F[X]\} . \tag{4}$$

In particular, the Lie derivative of X yields the component of V_r in the "direction" of X (other directions correspond to prolongation terms). Let

$$\Delta^{(r)}X = V_r\{X\} = \phi^{(r)}[X] \tag{5}$$

(where the functional $\phi^{(r)}$ may be local or nonlocal in X). Eq. (5) determines an infinitesimal change of X:

$$X' = X + a\phi^{(r)}[X] \tag{6}$$

(where a is an infinitesimal parameter). Clearly, $\phi^{(r)}$ must be *traceless* to preserve the sl(N, C)-valuedness of X. The symmetry condition in order that X' be a PSDYM solution, whenever X is a solution, is

$$\Delta^{(r)}G[X] = 0 , \tag{7}$$

where $G[X]$ is defined in eq. (3).

Eq. (7) may be written explicitly by using eq. (5) and the facts that (a) the Lie derivative is a derivation (i.e., satisfies the Leibniz rule), and (b) the Lie derivative with respect to a vertical vector field commutes with all products and powers of total derivatives. We thus obtain

$$\phi^{(r)}_{y\bar{y}} + \phi^{(r)}_{z\bar{z}} + [X_{\bar{z}}, \phi^{(r)}_{\bar{y}}] - [X_{\bar{y}}, \phi^{(r)}_{\bar{z}}] = 0 . \tag{8}$$

As we showed in ref. [1], the symmetry problems for the SDYM and PSDYM are intimately related. Specifically, any symmetry transformation for the PSDYM of the form (6) corresponds to a symmetry transformation for the SDYM of the form

$$J' = J + aJ\phi^{(r)}[X] . \tag{9}$$

Eq. (9) represents a genuine potential symmetry of the SDYM if $\phi^{(r)}$ contains terms (such as X, X_y, X_z, etc.) which are nonlocal in J. Conversely, any SDYM symmetry of the form $J' = J + aQ$ corresponds to a PSDYM symmetry $X' = X + aJ^{-1}Q$. We may thus state the general relation (in an obvious notation)

$$\Delta^{(r)}J = J\Delta^{(r)}X . \tag{10}$$

The symmetry condition (8) is a linear equation in $\phi^{(r)}$, for any PSDYM solution X. Consequently, for any given X, the sl(N, C)-valued solutions $\phi^{(r)}$ of eq. (8) form a linear space which we call S_X. A *recursion operator* $\hat{R}$ for the PSDYM is a linear (integro-differential) operator which maps the space S_X into itself. A *symmetry operator* $\hat{L}$ for the PSDYM is a linear operator which maps the set of all sl(N, C) PSDYM solutions into S_X. We note that any power $\hat{R}^n$ of a recursion operator is also a recursion operator, while the product $\hat{R}\hat{L}$ of a recursion operator and a symmetry operator is a symmetry operator. Thus, $\hat{R}^n\hat{L}X$ is a member of S_X.

We introduce the covariant derivative operators

$$\hat{A}_y \equiv D_y + [X_{\bar{z}},\ ] , \quad \hat{A}_z \equiv D_z - [X_{\bar{y}},\ ] .$$

With the aid of the Jacobi identity and the PSDYM equation, the zero-curvature condition $[\hat{A}_y, \hat{A}_z] = 0$ is shown to be satisfied. The linear operators $\hat{A}_y$ and $\hat{A}_z$ are derivations on the algebra of Sl(N, C)-valued functions. (We note that the Leibniz rule is written

$$\hat{A}_y[F, G]=[\hat{A}_yF, G]+[F, \hat{A}_yG]$$

(and similarly for $\hat{A}_z$), where F and G are arbitrary matrix functions.) According to the symmetry condition (8), the space S_X is the subspace of the above-mentioned algebra on which the following local operator equation is satisfied,

$$\hat{A}_y D_{\bar{y}}+\hat{A}_z D_{\bar{z}} \equiv D_{\bar{y}}\hat{A}_y+D_{\bar{z}}\hat{A}_z=0 . \tag{11}$$

Integrating eq. (11) with respect to $\bar{y}$ and $\bar{z}$ we obtain an equivalent nonlocal operator equation,

$$D_{\bar{z}}^{-1}\hat{A}_y+D_{\bar{y}}^{-1}\hat{A}_z=0 , \tag{12}$$

which is valid on S_X.

We consider the linear nonlocal operator

$$\hat{R}=D_{\bar{z}}^{-1}\hat{A}_y . \tag{13}$$

Clearly, $\hat{R}$ maps the algebra of sl(N, C)-valued functions into itself (it is *not*, however, a derivation on this algebra). We now show that $\hat{R}$ also maps the subspace S_X into itself, i.e., is a *recursion operator* for the PSDYM. Indeed, we simply note that, on S_X,

$$(\hat{A}_y D_{\bar{y}}+\hat{A}_z D_{\bar{z}})\hat{R}=[\hat{A}_z, \hat{A}_y]=0 ,$$

where we have used eq. (12). Hence, according to eq. (11), the space $\hat{R}S_X$ is a subspace of S_X.

The Lie derivative and the recursion operator satisfy the commutation relation

$$[\Delta^{(r)}, \hat{R}]=D_{\bar{z}}^{-1}[D_{\bar{z}}\Delta^{(r)}X,\ \ ] , \tag{14}$$

which is easily verified by using the previously mentioned properties of $\Delta^{(r)}$. More generally, the following relation is valid,

$$[\Delta^{(r)}, \hat{R}^n]F = \sum_{k=1}^{n} \hat{R}^{k-1}D_{\bar{z}}^{-1}[D_{\bar{z}}\Delta^{(r)}X, \hat{R}^{n-k}F] , \tag{15}$$

where $n=1, 2, 3, ...$, and F is an arbitrary matrix function.

Although $\hat{R}$ is not a derivation, it satisfies a sort of generalized "Leibniz rule" which follows directly from the derivation property of $\hat{A}_y$:

$$\hat{R}[F, G]=[\hat{R}F, G]+[F, \hat{R}G] -D_{\bar{z}}^{-1}([\hat{R}F, G_{\bar{z}}]+[F_{\bar{z}}, \hat{R}G]) . \tag{16}$$

From this, one may deduce the more general relation

$$D_{\bar{z}}^{-1}[D_{\bar{z}}\hat{R}^nF, G] =D_{\bar{z}}^{-1}[F_{\bar{z}}, \hat{R}^nG]+ \sum_{k=1}^{n} \hat{R}[\hat{R}^{n-k}F, \hat{R}^{k-1}G] - \sum_{k=1}^{n} [\hat{R}^{n-k}F, \hat{R}^kG] . \tag{17}$$

Finally, we state the following general property.

Let $\hat{L}$ be a linear operator satisfying the commutation relation

$$[\hat{L}, \hat{R}]=D_{\bar{z}}^{-1}[D_{\bar{z}}\hat{L}X,\ \ ]-\lambda\hat{R} , \tag{18}$$

where λ is a constant. Then, for any matrix function F,

$$[\hat{L}, \hat{R}^n]F= \sum_{k=1}^{n} \hat{R}^{k-1}D_{\bar{z}}^{-1}[D_{\bar{z}}\hat{L}X, \hat{R}^{n-k}F]-n\lambda\hat{R}^nF . \tag{19}$$

The (somewhat lengthy) formalism developed above serves a main purpose: to enable one to formulate and prove the two theorems that follow. The proofs of the theorems are long and cannot be given here in full. Their reproduction, however, is relatively straightforward, the only essential difficulty being the large number of necessary series manipulations.

Theorem 1. Consider the infinite set of transformations

$$\Delta_k^{(n)}X=\hat{R}^n\hat{L}_kX \quad (n=0, 1, 2, ..., k=1, 2, ... p) , \tag{20}$$

where $\hat{R}$ is the recursion operator (13) and the $\hat{L}_k$ are symmetry operators obeying the commutation relations

$$[\hat{L}_i, \hat{L}_j]=-C_{ij}^k\hat{L}_k , \tag{21}$$

$$[\Delta_i^{(n)}, \hat{L}_k]=0 \quad (\text{all } i, k) , \tag{22}$$

$$[\hat{L}_k, \hat{R}]=D_{\bar{z}}^{-1}[D_{\bar{z}}\hat{L}_kX,\ \ ] \quad (\text{on } S_X) \tag{23}$$

(where the last relation is generally valid only on S_X).

The the set of transformations (20) is a Kac–Moody algebra associated with the p-dimensional Lie algebra (21) generated by $\hat{L}_k$:

$$[\Delta_i^{(m)}, \Delta_j^{(n)}]X=C_{ij}^k\Delta_k^{(m+n)}X . \tag{24}$$

Theorem 2. Consider the infinite set of transformations

$$\Delta^{(n)}X=\hat{R}^n\hat{L}X \quad (n=0,1,2,...)\,, \tag{25}$$

where $\hat{L}$ is a symmetry operator obeying the commutation relations

$$[\Delta^{(n)},\hat{L}]=0\,, \tag{26}$$

$$[\hat{L},\hat{R}]=D_{\bar{z}}^{-1}[D_{\bar{z}}\hat{L}X,\ ]-\hat{R} \quad (\text{on } S_X) \tag{27}$$

(where the last relation is generally valid only on S_X).

Then, to within a sign, the set of transformations (25) is a Virasoro algebra:

$$[\Delta^{(m)},\Delta^{(n)}]X=(n-m)\Delta^{(m+n)}X\,. \tag{28}$$

To prove theorem 1, we write

$$\Delta_i^{(m)}\Delta_j^{(n)}X=\Delta_i^{(m)}\hat{R}^n\hat{L}_jX$$
$$=\hat{R}^n\Delta_i^{(m)}\hat{L}_jX+[\Delta_i^{(m)},\hat{R}^n]\hat{L}_jX$$

and, using eq. (22),

$$\Delta_i^{(m)}\hat{L}_jX=\hat{L}_j\Delta_i^{(m)}X=\hat{L}_j\hat{R}^m\hat{L}_iX$$
$$=\hat{R}^m\hat{L}_j\hat{L}_iX+[\hat{L}_j,\hat{R}^m]\hat{L}_iX\,.$$

The two commutators that appear in the above relations are calculated with the aid of eqs. (15) and (19). In the latter equation we put $\lambda=0$, as follows by comparing eqs. (18) and (23). We thus find a relation of the form

$$\Delta_i^{(m)}\Delta_j^{(n)}X=\hat{R}^{m+n}\hat{L}_j\hat{L}_iX+K(m,i;n,j)\,.$$

Application of the generalized Leibniz rule (17) (in combination with necessary partial integrations) yields the result that the expression K (which is actually a sum of series) is symmetric under simultaneous exchanges of m and i with n and j, respectively. Thus we finally have

$$[\Delta_i^{(m)},\Delta_j^{(n)}]X=\hat{R}^{m+n}[\hat{L}_j,\hat{L}_i]X\,.$$

The result (24) follows directly by using eqs. (21) and (20).

The proof of theorem 2 is similar. The main difference is that we must now use eq. (19) with $\lambda=1$, as follows by comparing eqs. (18) and (27).

Having the two fundamental theorems at our disposal, we are now in a position to draw important conclusions regarding the symmetry group of the PSDYM, thus also of the SDYM. To begin with, we need to find a set of symmetry operators for the PSDYM. These are obtained by verticalizing the point symmetries of this equation. We have derived the isogroup of point transformations of the PSDYM (details to appear elsewhere) by using the generalized isovector methods developed by this author and B.K. Harrison [9–11]. By verticalizing these symmetries we obtain a set of first-order Lie–Bäcklund transformations, certain of which are local in J (hence are not potential symmetries in the strict sense) while the remaining ones form a group of transformations which are nonlocal in J. These latter transformations are expressible in the form

$$\Delta_a^{(0)}X=\hat{L}_aX\,,$$

where $\hat{L}_a$ are the corresponding symmetry operators. Application of the recursion operator then yields an infinite group of nonlocal symmetries,

$$\Delta_a^{(n)}X=\hat{R}^n\hat{L}_aX \quad (n=0,1,2,...)\,. \tag{29}$$

Here is now a description of the set $\{\hat{L}_a\}$.

(A) *Internal symmetries.* Let $\{T_k\}$ be a basis for $\mathrm{sl}(N,\mathrm{C})$:

$$[T_i,T_j]=C_{ij}^kT_k\,.$$

Define the linear operators $\hat{L}_k$ by

$$\hat{L}_kM=[M,T_k]\,,$$

where M is any $\mathrm{sl}(N,\mathrm{C})$ matrix. These operators are symmetry operators for the PSDYM which satisfy the conditions of theorem 1, i.e., eqs. (21)–(23). Therefore, the infinite set of transformations of the form (29),

$$\Delta_k^{(n)}X=\hat{R}^n\hat{L}_kX=\hat{R}^n[X,T_k]\,, \tag{30}$$

is a Kac–Moody algebra associated with $\mathrm{sl}(N,\mathrm{C})$. This is precisely the familiar "hidden" symmetry of the SDYM [5,12] which is now recognized to be a potential symmetry.

(B) *Symmetries in the base space.* We now turn to symmetry operators which correspond to point transformations in the space of the independent variables x^μ. Potential symmetries nonlocal in J are expressed by the following operators,

$$\hat{L}_1=D_y\,,\quad \hat{L}_2=D_z\,,\quad \hat{L}_3=zD_y-\bar{y}D_{\bar{z}}\,,$$

$$\hat{L}_4 = yD_z - \bar{z}D_{\bar{y}}\,, \quad \hat{L}_5 = yD_y - zD_z - \bar{y}D_{\bar{y}} + \bar{z}D_{\bar{z}}\,,$$
$$\hat{L}_6 = 1 + yD_y + zD_z\,, \quad \hat{L}_7 = 1 - \bar{y}D_{\bar{y}} - \bar{z}D_{\bar{z}}\,,$$
$$\hat{L}_8 = y\hat{L}_6 + \bar{z}(yD_{\bar{z}} - zD_{\bar{y}})\,,$$
$$\hat{L}_9 = z\hat{L}_6 + \bar{y}(zD_{\bar{y}} - yD_{\bar{z}})\,.$$

The $\hat{L}_1$, $\hat{L}_2$ represent translations of y and z, respectively, while the $\hat{L}_3$, $\hat{L}_4$ represent "rotational" symmetries. The $\hat{L}_5$, $\hat{L}_6$, $\hat{L}_7$ express scale transformations. In particular, $\hat{L}_5$ is a scale change that involves only the x^μ, whereas $\hat{L}_6$ and $\hat{L}_7$ contain scale changes of X as well. Finally, $\hat{L}_8$ and $\hat{L}_9$ represent nonlinear transformations of the x^μ. Presumably, these last two operators reflect the special conformal invariance of the covariant SDYM equations [11]. Note that all nine operators produce sl(N, C) symmetries.

The first five operators $\{\hat{L}_1, ..., \hat{L}_5\}$ form a Lie algebra which we call $\underset{\sim}{g}$. Moreover, these operators satisfy the conditions of theorem 1. Therefore, the infinite set of transformations of the form (29)

$$\Delta_k^{(n)} X = \hat{R}^n \hat{L}_k X \quad (k=1, ..., 5) \tag{31}$$

is a *Kac–Moody* algebra associated with $\underset{\sim}{g}$. (We make the technical observation that $\underset{\sim}{g}$ is the semidirect sum of the Abelian ideal $\{\hat{L}_1, \hat{L}_2\}$ and the subalgebra $\{\hat{L}_3, \hat{L}_4, \hat{L}_5\}$. In particular, the algebra $\underset{\sim}{g}$ is not semisimple.)

On the other hand, the operators $\hat{L}_6$ and $\hat{L}_7$ *separately* satisfy the requirements of theorem 2. Thus we have a double infinity of symmetries of the form

$$\Delta^{(n)} X = \hat{R}^n \hat{L} X\,, \quad \hat{L} \equiv \hat{L}_6 \text{ or } \hat{L}_7\,, \tag{32}$$

where each sequence is a *Virasoro* algebra.

The presence of the symmetry operators $\hat{L}_8$ and $\hat{L}_9$ adds more structure to the problem. Indeed, these operators do not conform to the conditions of either theorem 1 or 2. The understanding of this additional structure requires further investigation, the results of which will be reported in future articles.

As we noted above, all of our symmetries preserve the sl(N, C)-valuedness of PSDYM solutions, or, equivalently, the SL(N, C)-valuedness of SDYM solutions. Physical SDYM solutions, however, are also Hermitian and this Hermiticity must be preserved by symmetry transformations. This is accomplished by replacing eq. (10) with the following one,

$$\Delta^{(r)} J = J\Delta^{(r)} X + (\Delta^{(r)} X)^\dagger J \tag{33}$$

(where the dagger denotes Hermitian conjugation). Eq. (33) represents an SDYM symmetry whenever the condition $J^\dagger = J$ is satisfied in real space. Clearly, then, the transformation (33) preserves the Hermiticity of J. Moreover, this transformation also preserves the condition $\det J = 1$, as follows from the observation that

$$\operatorname{tr}(J^{-1}\Delta^{(r)} J) = 0 \quad \text{when } \operatorname{tr}\Delta^{(r)} X = 0\,.$$

Hence, our symmetry transformations are suitable for producing real SU(N) solutions.

In conclusion, the introduction of potential symmetries has significantly enhanced the symmetry group of the SDYM. In this paper the complete set of potential symmetries was presented and their Lie algebraic structure was studied. The potential symmetries associated with point transformations in the fiber space (internal symmetries) constitute the familiar Kac–Moody "hidden" symmetry of the SDYM, a fact that was first pointed out in ref. [1]. The present article shows that the symmetries in the base space (coordinate symmetries) are also associated with infinite-dimensional Lie algebraic structures of familiar types. We have uncovered a Kac–Moody algebra related to a five-parameter Lie algebra of coordinate transformations, as well as the double presence of a Virasoro algebra. All transformations conform to the usual physical conditions imposed on SDYM solutions.

The presence of current algebras in SDYM theory is presumably another manifestation of the total integrability of the SDYM. In particular, this nonlinear system is now seen to possess infinite sets of commuting symmetries, each set being the subalgebra of the Kac–Moody algebra (31) that corresponds to a fixed value of the index k. This situation is typical with most integrable systems. In addition to their mathematical importance, we expect that these symmetry structures will provide useful new information regarding the physics of SDYM fields, in general. One thing that needs to be checked in this connection is the persistence or not of these structures in the quantized version of the theory. Another interesting question is whether these symmetries are indeed of variational nature, as conjectured previously [1]. A closer examination of these and other related matters will be the subject of future papers.

The reader may wonder whether the infinite set of potential symmetries reported here is exhaustive, in view of the fact that our algebra was generated by using only first-order Lie–Bäcklund transformations (point symmetries) as a basis. As argued previously [10] it appears that the SDYM and, likewise, the PSDYM do not possess higher-order *local* (Lie–Bäcklund) symmetries (there is no rigorous proof of this statement, however). This is presumably related to the apparent nonexistence of a *local* recursion operator for these equations. Finally, we remark that our infinite set of symmetries can actually be doubled by inverting the recursion operator. (The invertibility, in principle, of this operator follows from the results of ref. [1].) The Lie algebraic structure of this enlarged symmetry set is currently under investigation.

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Appendix

Let us briefly outline the isovector method [9–11] used to derive the point symmetry group of the PSDYM (detailed calculations will appear elsewhere). This systematic technique guarantees a complete solution to the aforementioned symmetry problem.

Consider the system of first-order partial differential equations

$$\psi_y-\phi_z+[\phi,\psi]=0\,, \tag{A.1a}$$

$$\phi_{\bar{y}}+\psi_{\bar{z}}=0\,. \tag{A.1b}$$

The substitutions $\phi=X_{\bar{z}}$, $\psi=-X_{\bar{y}}$, reduce eqs. (A.1a) and (A.1b) to the PSDYM equation (3). Eqs. (A.1a) and (A.1b) may be represented by a pair of differential four-forms in a space of six variables:

$$\gamma_1=\mathrm{d}\psi\,\mathrm{d}z\,\mathrm{d}\bar{y}\,\mathrm{d}\bar{z}-\mathrm{d}y\,\mathrm{d}\phi\,\mathrm{d}\bar{y}\,\mathrm{d}\bar{z}+[\phi,\psi]\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\bar{y}\,\mathrm{d}\bar{z}\,,$$

$$\gamma_2=\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\phi\,\mathrm{d}\bar{z}+\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\bar{y}\,\mathrm{d}\psi\,,$$

The conditions $\gamma_1=0$, $\gamma_2=0$, are equivalent to eqs. (A.1a) and (A.1b). It may be verified that γ_1 and γ_2 generate a differential ideal of four-forms.

The point symmetries of system (A.1) may be expressed as isovector fields in the variables $\{x^\mu,\phi,\psi\}$, which satisfy the following requirement: the Lie derivative with respect to an isovector maps the ideal of γ_1 and γ_2 into itself. If V is such a vector, the following condition is satisfied,

$$\Delta^{(V)}\gamma_i=\xi_i^k\gamma_k+M_i^k\gamma_k+\gamma_k N_i^k \tag{A.2}$$

($i=1$, 2), where the ξ_i^k are scalar functions while M_i^k and N_i^k are matrix-valued. Eqs. (A.2) are solved for the components of the isovectors V. Given that ϕ and ψ are dependent upon X, these vectors can be re-expressed in terms of the variables $\{x^\mu, X\}$. Furthermore, the isovectors may be rewritten in vertical form, as follows,

$$V_k=(\hat{L}_k X)\mathrm{D}_X+\text{prolongation terms}\,.$$

The set $\{\hat{L}_k\}$ contains the linear symmetry operators mentioned earlier, plus three more: $\hat{L}=\mathrm{D}_{\bar{y}}$, $\hat{L}=D_{\bar{z}}$ and $\hat{L}=z\mathrm{D}_y-y\mathrm{D}_z$. These latter operators are discarded, however, since they yield symmetries which are local in J (cf. eqs. (2)).

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LETTER TO THE EDITOR

Lax pair, hidden symmetries, and infinite sequences of conserved currents for self-dual Yang–Mills fields

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Abstract. A Lax pair which linearizes the self-dual Yang–Mills (SDYM) equation is found and shown to be intimately related to the general symmetry problem for SDYM. The linear system is used to derive an invertible recursion operator that produces new infinite sequences of non-local symmetries and associated conservation laws for SDYM.

The integrability properties of the self-dual Yang–Mills (SDYM) equation have been a subject of extensive study over the past fifteen years. As is well known, this nonlinear equation, when properly formulated, displays many of the typical characteristics of an 'integrable' system, such as parametric Bäcklund transformations [1–4], infinite sequences of conservation laws, both non-local [5–7] and local [8], linear system (Lax pair) [9, 10, 6], Painlevé property [11, 12], etc. In particular, the Lax pair was shown to be related both to the presence of a Kac–Moody 'hidden' symmetry [13–15] and to the existence of an infiite number of non-local conserved currents [10].

This letter makes the observation that the SDYM equation can be linearized in more than one way. We propose a new Lax pair for SDYM which allows the relationship between the symmetry and integrability aspects of this equation to become most transparent. This Lax pair is used to construct an invertible recursion operator which produces new infinite sequences of non-local symmetries and associated conservation laws for SDYM. The previously mentioned Kac–Moody symmetry appears naturally as a subsymmetry generated by purely internal transformations.

We write the SDYM in gauge-invariant form [16, 17, 6]:

$$F(J) \equiv D_{\bar{y}}(J^{-1}J_y) + D_{\bar{z}}(J^{-1}J_z) = 0 \qquad (1)$$

(where we use the notation $J_y = D_y J \equiv \partial J/\partial y$, etc, for partial derivatives). The variables $y, z, \bar{y}, \bar{z}$ are constructed from the coordinates of an underlying complexified Euclidean space in such a way that $\bar{y}$ and $\bar{z}$ become the complex conjugates of y and z, respectively, when the above space is real. The variable J is, in general, an N-dimensional complex, non-singular matrix. For real SU(N) gauge theory, J is required to be a Hermitian SL(N, C) matrix in real space.

Let $J' = J + \alpha Q(J)$ be an infinitesimal symmetry transformation, i.e. one which leaves equation (1) invariant. Here $Q(J)$ is a functional which may be local or non-local

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in J, while α is an infinitesimal parameter. The symmetry condition in order that $F(J')=0$, whenever $F(J)=0$, is

$$D_{\bar{y}}\{J^{-1}[Q(J)J^{-1}]_y J\}+D_{\bar{z}}\{J^{-1}[Q(J)J^{-1}]_z J\}=0. \tag{2}$$

One often says that the functional $Q(J)$ is a symmetry characteristic for (1).

Equation (2) has been solved for the particularly simple case of point symmetries by using isovector techniques [2, 3]. Moreover, the internal symmetry: $Q(J)=JM$, where M is a constant matrix, serves as a basis for constructing the Kac–Moody 'hidden' symmetry of SDYM [13–15]. We will presently extend the invariance group by adding infinite sequences of symmetries associated with coordinate transformations. To begin with, we propose the following linearization of SDYM.

Proposition 1. Consider the pair of linear equations for ψ:

$$J(J^{-1}\psi)_{\bar{z}}=\lambda(\psi J^{-1})_y J \qquad J(J^{-1}\psi)_{\bar{y}}=-\lambda(\psi J^{-1})_z J \tag{3}$$

where λ is a complex parameter and J is a matrix function. This system is integrable for ψ if J is a solution of (1): $F(J)=0$. Moreover, if $\psi(J;\lambda)$ is a solution of the linear system (3), for some SDYM field J, then ψ is a symmetry characteristic, i.e. satisfies (2).

Proof. The integrability condition $(J^{-1}\psi)_{\bar{z}\bar{y}}=(J^{-1}\psi)_{\bar{y}\bar{z}}$ yields

$$D_{\bar{y}}[J^{-1}(\psi J^{-1})_y J]+D_{\bar{z}}[J^{-1}(\psi J^{-1})_z J]=0. \tag{4}$$

The integrability condition $\psi_{yz}=\psi_{zy}$ yields (after a lengthy calculation, and by using (4)):

$$[J^{-1}\psi, F(J)]=0.$$

For this to be satisfied independently of ψ, one must have $F(J)=0$. A comparison of (4) and (2) then implies that $\psi(J;\lambda)$ is a symmetry characteristic of (1).

Thus, equations (3) constitute a Lax pair for SDYM, the solution ψ of which pair is a symmetry generator. It is natural to seek an explicit construction of ψ for given J and λ. To this end, we try a Laurent expansion in powers of the parameter λ:

$$\psi(J;\lambda)=\sum_{n=-\infty}^{+\infty}\lambda^n Q^{(n)}(J). \tag{5}$$

Substituting this into equations (3), and equating the coefficients of λ^{n+1}, we obtain the pair of equations:

$$J[J^{-1}Q^{(n+1)}]_{\bar{z}}=[Q^{(n)}J^{-1}]_y J \qquad J[J^{-1}Q^{(n+1)}]_{\bar{y}}=-[Q^{(n)}J^{-1}]_z J. \tag{6}$$

The consistency of these relations requires that both $Q^{(n)}$ and $Q^{(n+1)}$ satisfy (2). Technically speaking, equations (6) are a strong Bäcklund transformation for the symmetry condition (2) of SDYM, for a given solution J of (1). Equations (6) may be rewritten in the form of an invertible non-local recursion operator:

$$Q^{(n+1)}=JD_{\bar{z}}^{-1}\{J^{-1}[Q^{(n)}J^{-1}]_y J\} \qquad Q^{(n-1)}=-D_{\bar{z}}^{-1}\{J[J^{-1}Q^{(n)}]_{\bar{y}}J^{-1}\}J. \tag{7}$$

Starting with a known symmetry $Q^{(0)}(J)$ of SDYM (say, a local symmetry), one may construct an infinite sequence of symmetries $Q^{(n)}(J)$ (where $n=\pm1,\pm2,\pm3,\ldots,\pm\infty$) simply by employing the recursion relations (7). At the same time, the solution (5) of the Lax pair is formally represented as an infinite sum of symmetry characteristics of SDYM.

If the original (untransformed) solution J satisfies $\det J = 1$ and $J^{\dagger} = J$ in real space, the conditions in order that a symmetry $Q(J)$ preserve these properties of J, are $\operatorname{tr}(J^{-1}Q) = 0$ and $Q^{\dagger} = Q$ in real space (where the dagger denotes Hermitian conjugation). Let $Q^{(n)}$ be a characteristic with these properties. In general, neither $Q^{(n+1)}$ nor $Q^{(n-1)}$, as given by equations (7), will be Hermitian. To take care of this problem, we use the fact that the symmetry condition (2) is linear in $Q(J)$, hence the difference of two solutions is again a solution (for the same J). Thus we consider the following recursion relation in place of those of equations (7):

$$Q^{(n+1)} = JD_{\bar{z}}^{-1}\{J^{-1}[Q^{(n)}J^{-1}]_y J\} + D_z^{-1}\{J[J^{-1}Q^{(n)}]_{\bar{y}}J^{-1}\}J. \tag{8}$$

It is readily verified that this operator preserves the required properties of $Q^{(n)}$ for Hermitian SL(N, C) SDYM solutions.

The recursion operator does more than produce new symmetries. Returning to the symmetry condition (2) we observe that it has the form of a continuity equation which is satisfied for all symmetry characteristics $Q^{(n)}(J)$:

$$D_{\bar{y}}\{J^{-1}[Q^{(n)}(J)J^{-1}]_y J\} + D_{\bar{z}}\{J^{-1}[Q^{(n)}(J)J^{-1}]_z J\} = 0. \tag{9}$$

We thus obtain an infinite sequence of non-local conservation laws for SDYM, corresponding to the infinite sequence of non-local characteristics $Q^{(n)}(J)$. We note that the conserved 'charges' are linearly dependent upon symmetry characteristics. This feature is new, not present in older conservation laws for SDYM [5, 7], and may suggest that these currents are associated with some underlying Noether structure.

We now study the relationship of our Lax pair (3) to the one known previously [6, 9, 10] for SDYM:

$$X_{\bar{z}} = \lambda(X_y + J^{-1}J_y X) \qquad X_{\bar{y}} = -\lambda(X_z + J^{-1}J_z X). \tag{10}$$

We have found a simple algebraic relation which allows one to construct solutions ψ of (3) from solutions X of (10) (but not vice versa) for the same J:

Proposition 2. Let $X(J; \lambda)$ be a solution of equations (10), for a given SDYM solution J. Consider the function $\psi(J; \lambda)$ defined by

$$\psi = JXTX^{-1} \tag{11}$$

where

$$T = f(y + \lambda\bar{z}, z - \lambda\bar{y}, \lambda) \tag{12}$$

is an arbitrary function of the indicated variables. Then, ψ is a solution of equations (3).

Proof. We first note that, according to (12), T satisfies the relations $T_{\bar{z}} = \lambda T_y$ and $T_{\bar{y}} = -\lambda T_z$. Putting $\phi \equiv XTX^{-1}$, and using equations (10), we find that ϕ satisfies the pair of equations

$$\phi_{\bar{z}} = \lambda(\phi_y + [J^{-1}J_y, \phi]) \qquad \phi_{\bar{y}} = -\lambda(\phi_z + [J^{-1}J_z, \phi]).$$

By substituting $\phi = J^{-1}\psi$, we recover the linear system (3) for ψ.

Thus, (11) and (12) constitute a weak, non-auto-Bäcklund transformation which produces solutions of the Lax pair (3) from solutions of the Lax pair (10) (this does *not* imply, however, that *all* solutions of (3) may be obtained in this way). This transformation is of practical value when seeking solutions of (3), considering the fact that several solutions of (10) are known (see, for example, [9] and [10] for results related to the multi-instanton solution). Special solutions ψ of the Lax pair (3) are important since, as we have seen, they yield new hidden symmetries and conservation laws for SDYM.

In concluding this letter, we give examples of new symmetries by constructing a few of them explicitly. The conditions $\det J = 1$ and $J^\dagger = J$ will be assumed throughout.

(1) First, we remark that the known symmetries can be recovered by using our symmetry-generating process. Let us start with the internal symmetry $Q^{(0)}(J) = JM + M^\dagger J$, where M is a constant, traceless matrix. Application of the recursion operator (8) yields, after a straightforward calculation

$$Q^{(1)}(J) = J[P, M] + [M^\dagger, \bar{P}]J$$

where P and $\bar{P}$ are potentials for the SDYM equation, defined by $J^{-1}J_y = P_{\bar{z}}$, $J^{-1}J_z = -P_{\bar{y}}$ and $J_{\bar{y}}J^{-1} = \bar{P}_z$, $J_{\bar{z}}J^{-1} = -\bar{P}_y$ (note that, by the conditions imposed on J, the P and $\bar{P}$ are traceless and Hermitian-conjugately related in *real* space).

Repeated application of the recursion operator, and expansion of the matrix M in the basis of $\mathrm{sl}(N, C)$, yield an infinite set of infinitesimal transformations which constitute the familiar Kac–Moody symmetry of SDYM [13–15]. In the literature [13] this symmetry was found by exploiting the infinitesimal transformation $\delta J = -JXMX^{-1}$, where X is a solution of system (10) and M is an infinitesimal constant matrix. (The connection of the aforementioned transformation with (11) is evident.)

(2) Let us start with the translational symmetry [3] $Q^{(0)}(J) = J_y + J_{\bar{y}}$ (note that $\mathrm{tr}(J^{-1}J_y) = 0$, etc). Application of the recursion operator (8) yields

$$Q^{(1)}(J) = J(P_y + P_{\bar{y}}) + (\bar{P}_y + \bar{P}_{\bar{y}})J$$

and so forth. We thus obtain an infinite sequence of new non-local symmetries and conservation laws; the latter are found by direct substitution of the $Q^{(n)}$ into (9).

(3) The dilational symmetry $Q^{(0)} = yJ_y + zJ_z + \bar{y}J_{\bar{y}} + \bar{z}J_{\bar{z}}$ yields

$$Q^{(1)} = J(yP_y + zP_z + \bar{y}P_{\bar{y}} + \bar{z}P_{\bar{z}}) + (y\bar{P}_y + z\bar{P}_z + \bar{y}\bar{P}_{\bar{y}} + \bar{z}\bar{P}_{\bar{z}})J$$

and so forth.

We work similarly for the remaining coordinate symmetries [2, 3]; i.e., the translational symmetry $Q^{(0)} = J_z + J_{\bar{z}}$, and the 'rotational' symmetry $Q^{(0)} = zJ_y - yJ_z + \bar{z}J_{\bar{y}} - \bar{y}J_{\bar{z}}$.

In summary, we have proposed a linearization of SDYM which makes the connection between symmetry and integrability most transparent. The Lax pair was used to construct an invertible recursion operator which, in turn, produced new hidden non-local symmetries and conservation laws. We have discussed possible representations for solutions of the Lax pair, either as infinite sums of symmetry characteristics, or as images, under a weak Bäcklund map, or solutions of the Belavin–Zakharov–Pohlmeyer–Chau linear system. The aforementioned map, being non-surjective, does not yield the general solution of the Lax pair; this probably explains why the older linear system fails to produce the complete symmetry group of SDYM, in contrast to the new one. The solution-generating aspects of the latter system will be explored in future publications.

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Lax pair, conserved currents, and hidden symmetry transformation for the Ernst equation

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Abstract

The matrix Ernst equation (a reduced form of the self-dual Yang–Mills equation) is written as the compatibility condition for solution of a linear "inverse scattering" system. This system is used to construct infinite sequences of nonlocal conserved charges, as well as an infinitesimal hidden symmetry transformation, for the Ernst equation.

The vacuum Einstein equations for stationary, axially symmetric gravitational fields, when formulated according to Ernst [1], exhibit a close relationship to the self-dual Yang–Mills (SDYM) equation in the Yang formulation [2–4]. Indeed, it has been shown [5–7] that the Ernst equation (as well as the Einstein–Maxwell equations) may be derived from SDYM by reduction, by imposing certain symmetry and real-valuedness conditions. Although solution-generating techniques for the Ernst equation have been known [8,9], our understanding of this equation is enhanced and new insights are gained by exploiting its relationship to SDYM.

Recently [10] a formulation of SDYM was proposed which unifies the symmetry and integrability properties of this equation. In principle, the method is based on the observation that the SDYM equation may be "linearized" in more than one way by different choices of a Lax pair. A particularly useful choice is one which incorporates both the equation of motion and its symmetry condition. By taking advantage of the conservation-law form of the latter condition, the Lax pair yields an infinite sequence of conserved currents for SDYM. Moreover, the solution of the Lax pair is a symmetry characteristic for SDYM.

Since the Ernst equation is derivable from SDYM, it is natural to expect that an analogous scheme, unifying symmetry and integrability properties, exists for this equation too. The purpose of this short communication is to present the main aspects of such a formulation of the Ernst equation. Our method consists in finding a Lax pair which lends itself, in a most straightforward way, to the construction of new hidden symmetries and infinite collections of nonlocal conserved charges. For brevity, proofs of various statements will only be outlined, leaving a fuller treatment to a future, more extensive article.

The Ernst equation, $(\mathrm{Re}\,E)\nabla^2 E=(\nabla E)^2$, may be conveniently placed in matrix form [5–7,11]:

$$F(g)\equiv(\rho g^{-1}g_\rho)_\rho+(\rho g^{-1}g_z)_z=0\,, \tag{1}$$

where subscripts indicate partial derivatives with respect to the variables ρ, z, collectively denoted x^μ

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($\mu = 1, 2$, respectively). The 2-dimensional matrix g is required to be real-valued, symmetric, and of unit determinant. In terms of the Ernst potential $E = f + i\omega$, the matrix g may be parametrized as follows,

$$g_{ik} = f^{-1} h_{ik} ,$$

$$h_{11} = 1, \quad h_{12} = h_{21} = \omega, \quad h_{22} = f^2 + \omega^2 .$$

In the sequel we will temporarily relax the restrictions on g mentioned before and return to them later, when conditions for new physical solutions of Eq. (1) are sought.

Eq. (1) can be re-stated in a very elegant as well as convenient, for our purposes, operator form. We introduce the linear operators

$$\hat{A}_\rho = \rho(\partial_\rho + [g^{-1} g_\rho, \ ]) ,$$

$$\hat{A}_z = \rho(\partial_z + [g^{-1} g_z, \ ])$$

(where $\partial_\rho = \partial/\partial\rho$, etc.). It is easily checked that the $\hat{A}_\rho$ and $\hat{A}_z$ are derivations on the Lie algebra of gl (2, $\mathbb{C}$)-valued functions (the Leibniz rule is expressed in terms of commutators of such functions). Furthermore, the following operator identity is satisfied:

$$[\hat{A}_\rho, \hat{A}_z] = \hat{A}_z \tag{2}$$

Proposition 1. The Ernst equation $F(g) = 0$ is equivalent to the operator equation

$$[\hat{A}_\rho, \partial_\rho] + [\hat{A}_z, \partial_z] + \partial_\rho = 0 \tag{3}$$

to be identically satisfied on any gl(2, $\mathbb{C}$)-valued function. That is, the above operator relation reduces to an identity on solutions g of Eq. (1).

Proof. If we let the left-hand side of Eq. (3) operate on an arbitrary gl(2, $\mathbb{C}$)-valued function $\psi(x^\mu)$, we get

$$[F(g), \psi] = 0 ,$$

which is satisfied independently of ψ if $F(g) = 0$.

We seek infinitesimal symmetry transformations $\delta g = \alpha Q$ (where α is an infinitesimal parameter and Q is a matrix function) which leave Eq. (1), thus also its operator counterpart (3), invariant in form. If we set $Q = g\phi$, then the general symmetry condition is expressed as follows,

$$(\partial_\rho \hat{A}_\rho + \partial_z \hat{A}_z)\phi = 0, \quad \text{whenever } F(g) = 0 . \tag{4}$$

Note that this has the form of a continuity equation with densities $\hat{A}_\rho\phi$ and $\hat{A}_z\phi$.

We now consider gl(2, $\mathbb{C}$)-valued functions $\psi(x^\mu, \lambda)$ depending on the (real) coordinates $x^\mu = \rho, z$ and on an auxiliary *complex* variable (parameter) λ. We require that ψ be single-valued and analytic (as a function of λ) in a deleted neighborhood D of the origin of the λ-plane (i.e., the origin itself excluded).

Proposition 2. Consider the linear system for $\psi(x^\mu, \lambda)$:

$$\hat{A}_\rho \psi - 2\lambda\psi_\lambda = (1/\lambda)\psi_z, \quad \hat{A}_z \psi = -(1/\lambda)\psi_\rho . \tag{5}$$

This system is integrable for ψ if g satisfies $F(g) = 0$.

Proof. The integrability condition $[\hat{A}_\rho, \hat{A}_z]\psi = \hat{A}_z\psi$, together with the obvious fact that $[\hat{A}_z, \partial_\lambda] = 0$, yield, after a lengthy calculation the following,

$$([\hat{A}_\rho, \partial_\rho] + [\hat{A}_z, \partial_z] + \partial_\rho)\psi = 0 ,$$

which is valid independently of ψ if the operator equation (3) is identically satisfied, i.e., if $F(g) = 0$.

One notes immediately the presence of derivatives with respect to the parameter λ in the Lax pair (5). This feature also appears in the Lax pair found previously by Belinski and Zakharov [12], although their equations are otherwise very different from ours. We note that no derivatives of this kind appear in the Lax pairs found for SDYM [10,13–15].

The linear system (5) has no nontrivial solution for $\lambda = 0$. Hence, we may assume that ψ is singular at the origin of the λ-plane, but otherwise regular in D. We may thus expand ψ into a Laurent series in D, for a given solution g of Eq. (1):

$$\psi(x^\mu, \lambda) = \sum_{n=-\infty}^{\infty} \lambda^n \phi^{(n)}(x^\mu) , \tag{6}$$

where

$$\phi^{(n)}(x^\mu) = \frac{1}{2\pi i} \int_C \frac{d\lambda}{\lambda^{n+1}} \psi(x^\mu, \lambda) \tag{7}$$

(here, C is a positively oriented simple closed contour around $\lambda = 0$, lying entirely in D). Clearly, the $\phi^{(n)}$ depend on the choice of g, since ψ does.

Proposition 3. The $\phi^{(n)}$ are conserved charges for Eq. (1).

Proof. Substituting Eq. (6) into the Lax pair (5), and equating coefficients of λ^n, we obtain the system of equations

$$(\hat{A}_\rho - 2n)\phi^{(n)} = \phi_z^{(n+1)}, \quad \hat{A}_z\phi^{(n)} = -\phi_\rho^{(n+1)} \tag{8}$$

($n=0, \pm 1, \pm 2, \ldots$). We may regard system (8) as a Bäcklund transformation (BT) relating $\phi^{(n)}$ and $\phi^{(n+1)}$ and depending parametrically on g. The integrability condition $[\partial_\rho, \partial_z]\phi^{(n+1)}=0$, yields the continuity equation

$$\partial_\rho[(\hat{A}_\rho - 2n)\phi^{(n)}] + \partial_z(\hat{A}_z\phi^{(n)}) = 0\,, \tag{9}$$

while the integrability condition $[\hat{A}_\rho, \hat{A}_z]\phi^{(n)} = \hat{A}_z\phi^{(n)}$, in combination with the equation of motion (1), yield a relation of the same form as Eq. (9) but with $(n+1)$ in place of n. Hence, Eq. (9) summarizes an infinite sequence of conservation laws for Eq. (1), valid for all $n=0, \pm 1, \pm 2, \ldots$. According to Eq. (8), the $\phi^{(n)}$ are *potentials* or conserved charges for these laws. These charges may be constructed recursively by repeated integration of the strong BT (8) in both "directions" (i.e., for both increasing and decreasing n), provided that $\phi^{(0)}$ is known.

Proposition 4. The charge $\phi^{(0)}$ satisfies the symmetry condition (4) whenever ψ is a solution to the Lax pair (5).

Proof. We simply note that Eq. (9) reduces to Eq. (4) for $n=0$. Alternatively, from Eq. (7) we have

$$\phi^{(0)}(x^\mu) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{\mathrm{d}\lambda}{\lambda}\, \psi(x^\mu, \lambda) \tag{10}$$

Substituting the above expression into Eq. (4), and using the Lax pair to eliminate $\hat{A}_\rho\psi$ and $\hat{A}_z\psi$, we get

$$(\partial_\rho \hat{A}_\rho + \partial_z \hat{A}_z)\phi^{(0)} = \frac{1}{\pi \mathrm{i}}\,\partial_\rho \int_C \psi_\lambda\, \mathrm{d}\lambda = 0\,,$$

in view of the fact that the primitive ψ of ψ_λ is, by assumption, a single-valued function of λ in D.

A recapitulation is helpful at this point. Given a solution g of Eq. (1) and a symmetry characteristic Q (in the sense that $\delta g = \alpha Q$ is a symmetry), the BT (8) allows one to construct the charges $\phi^{(n)}$ and hence to obtain a formal series representation (6) for the solution of the Lax pair (5). The recursive construction of the $\phi^{(n)}$ starts with $\phi^{(0)} = g^{-1}Q$, which satisfies condition (4). Conversely, if ψ is found by direct integration of the Lax pair for a given g, the $\phi^{(n)}$ are obtained from Eq. (7). In particular, $\phi^{(0)}$ satisfies Eq. (4) so that $Q = g\phi^{(0)}$ is a symmetry characteristic. We remark that, in contrast to the SDYM case [10,16], $\phi^{(n)}$ is not proportional to a symmetry characteristic for $n \neq 0$. Thus, the BT (8) is not a recursion operator for symmetries.

The infinitesimal transformation $\delta g = \alpha g \phi^{(0)}$, although a symmetry of the general matrix equation (1), does not by itself produce physical new solutions of the Ernst equation. Indeed, an admissible transformation $\delta g = \alpha Q$ must preserve the three constraints: $g^* = g$, $g^{\mathrm{T}} = g$, and $\det g = 1$ (an asterisk denotes complex conjugation, while T denotes matrix transposition), whenever these are assumed for the original solution g. The symmetry characteristic Q must in turn satisfy the following requirements: $Q^* = Q$, $Q^{\mathrm{T}} = Q$, and $\mathrm{tr}(g^{-1}Q) = 0$. These are generally not obeyed by $Q = g\phi^{(0)}$.

To circumvent this problem, we take advantage of the following properties: (1) If g satisfies Eq. (1), then so does g^{T}. Consequently, if $g^{\mathrm{T}} = g$, and if $\delta g = \alpha Q$ is a symmetry, then so is $\delta g = \alpha Q^{\mathrm{T}}$ as well as $\delta g = \alpha(Q + Q^{\mathrm{T}})$. (2) The Lax pair (5) is compatible with the constraints $\mathrm{tr}\,\psi = 0$ and $\psi(x^\mu, \lambda^*) = \psi^*(x^\mu, \lambda)$. Hence, we seek traceless solutions of (5) which take on real values when λ is restricted to the real axis.

The above observations suggest that we consider the following infinitesimal symmetry transformation,

$$\delta g = \frac{\alpha}{2\pi \mathrm{i}} \int_C \frac{\mathrm{d}\lambda}{\lambda}\, [g\psi(x^\mu, \lambda) + \psi^{\mathrm{T}}(x^\mu, \lambda) g]\,, \tag{11}$$

where g satisfies $g^* = g$, $g^{\mathrm{T}} = g$, and $\det g = 1$. It is readily verified that the corresponding symmetry characteristic Q conforms to the above-mentioned necessary physical conditions. (To verify $Q^* = Q$, we make the change of variable $\lambda^* = \xi$ in Q^*, replacing the contour C by its negative, and then change the contour back to C while factoring out a minus sign.)

Transformation (11), written concisely as $\delta g = \alpha Q[g]$, leads to one-parameter families of solutions $g(x^\mu;\epsilon)$ of Eq. (1), starting from seed solutions $g(x^\mu;0)$, by integration of the orbit equations for the (formal) vector field $V=Q[g]\partial/\partial g$. We have obtained a solution to these latter equations in the form of a power series in ϵ, the terms of which are constructed recursively. (Details will be given in the Appendix.)

To conclude this Letter, we give an example of using the BT (8) to obtain conserved charges for Eq. (1), for a given symmetry characteristic Q of this equation. We relax the physical constraints on Q since they are not essential to the validity of the conservation laws (the physical conditions on g, however, are still assumed to hold).

The symmetry $\delta g=\alpha gM$, where M is a constant matrix, corresponds to $Q=gM$ and $\phi^{(0)}=g^{-1}Q=M$. Integrating the BT (8) for $n=0$, we find $\phi^{(1)}=[X, M]$, where X is the potential of Eq. (1), defined by $\rho g^{-1}g_\rho=X_z$, $\rho g^{-1}g_z=-X_\rho$, and satisfying the equation

$$\rho(X_{\rho\rho}+X_{zz})-X_\rho+[X_z, X_\rho]=0 . \qquad (12)$$

Integrating the BT (8) for $n=1$, we find

$$\phi^{(2)}=[\Omega, M]+\tfrac{1}{2}[X,[X,M]] ,$$

where Ω is the potential of Eq. (12), satisfying

$$\rho X_\rho-2X+\tfrac{1}{2}[X_z, X]=\Omega_z ,$$

$$\rho X_z-\tfrac{1}{2}[X_\rho, X]=-\Omega_\rho .$$

Conserved charges $\phi^{(n)}$ with $n<0$ are also obtained from $\phi^{(0)}$ by integration of the BT (8) in the reverse direction (i.e., for $n=-1, -2, ...$). Further charges are similarly constructed for $Q=g_z$ (z-translation) and $Q=\rho g_\rho+zg_z$ (scale-change of the x^μ).

In summary, we have proposed a formulation of the Ernst equation which treats symmetry and integrability in a unified manner; a suitable Lax pair was found by means of which new hidden symmetries and infinite sequences of nonlocal conserved currents were discovered. Although our method is similar, in spirit, to the one used previously for the 4-dimensional SDYM equation [10], our results reveal some significant differences between the aforementioned equation and the Ernst equation. Specifically: (a) The Lax pair (5) contains derivatives with respect to the spectral parameter, which is not the case with the Lax pairs for SDYM. (b) The BT (8) depends explicitly on the index n and is not a recursion operator for symmetries, in contrast to the corresponding BT for SDYM. We attribute these differences mostly to the less symmetric form of the Ernst equation, compared to SDYM, due to the explicit presence of an independent variable in this equation.

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Appendix. We now briefly describe the recursive process for obtaining one-parameter families of solutions $g(\epsilon)$ of Eq. (1), starting from seed solutions $g(0)=g$.

Consider the initial-value problem:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}g(\epsilon)=Q[g(\epsilon)], \quad g(o)=g , \qquad (\mathrm{A.1})$$

where, in general, $Q[g]$ is given by

$$Q=\frac{1}{2\pi \mathrm{i}}\int_C \frac{\mathrm{d}\lambda}{\lambda}\,[g\psi(x^\mu,\lambda)+\psi^{\mathrm{T}}(x^\mu,\lambda)g] . \qquad (\mathrm{A.2})$$

We seek solutions in the form

$$g(\epsilon)=\sum_{n=0}^{\infty}\epsilon^n g^{(n)}, \quad g^{(0)}=g . \qquad (\mathrm{A.3})$$

Given that the solution ψ of the Lax pair (5) depends implicitly on g, we may also write

$$\psi(\epsilon)=\sum_{n=0}^{\infty}\epsilon^n\psi^{(n)}, \quad \psi^{(0)}=\psi , \qquad (\mathrm{A.4})$$

where ψ and $\psi(\epsilon)$ are solutions of Eq. (5) corresponding to g and $g(\epsilon)$, respectively. Finally, let us set

$$g^{-1}(\epsilon)=\sum_{n=0}^{\infty}\epsilon^n h^{(n)} . \qquad (\mathrm{A.5})$$

It is readily verified that $h^{(n)}$ is fully determined by the $g^{(0)}, g^{(1)}, ..., g^{(n)}$. For example,

$$h^{(0)}=g^{-1} ,$$

$$h^{(1)}=-g^{-1}g^{(1)}g^{-1} ,$$

$$h^{(2)}=g^{-1}g^{(1)}g^{-1}g^{(1)}g^{-1}-g^{-1}g^{(2)}g^{-1} ,$$

etc.

Substituting Eqs. (A.3)–(A.5) into the linear system (5), and equating coefficients of ϵ^n, we find a system of equations by which $\psi^{(n)}$ is obtained by integration once the $g^{(0)}, ..., g^{(n)}$ and $\psi^{(0)}, ..., \psi^{(n-1)}$ have been determined:

$$\rho\left(\psi_\rho^{(n)} + \sum_{l=0}^{n}\sum_{m=0}^{l}[h^{(m)}g_\rho^{(l-m)}, \psi^{(n-l)}]\right) - 2\lambda\psi_\lambda^{(n)} - (1/\lambda)\psi_z^{(n)} = 0\,,$$

$$\rho\left(\psi_z^{(n)} + \sum_{l=0}^{n}\sum_{m=0}^{l}[h^{(m)}g_z^{(l-m)}, \psi^{(n-l)}]\right) + (1/\lambda)\psi_\rho^{(n)} = 0\,. \qquad (A.6)$$

Now, the expansion of Eqs. (A.1), (A.2) in ϵ yields a recursion relation by which $g^{(n+1)}$ is constructed once the $g^{(0)}, ..., g^{(n)}$ and $\psi^{(0)}, ..., \psi^{(n)}$ are known:

$$g^{(n+1)} = \frac{1}{2(n+1)\pi i} \times \sum_{m=0}^{n}\int_C \frac{d\lambda}{\lambda}\left(g^{(m)}\psi^{(n-m)} + [\psi^{(m)}]^T g^{(n-m)}\right). \qquad (A.7)$$

Thus, starting with a seed solution g, we may construct the $g^{(n)}$ of Eq. (A.3) recursively by using Eqs. (A.6) and (A.7). This constitutes a formal solution to the problem (A.1), (A.2).

Approximate or perturbative solutions may be obtained for small values of the parameter ϵ. The possible physical significance of such solutions will be explored in future works.

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A Method for Constructing a Lax Pair for the Ernst Equation

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Abstract: A systematic construction of a Lax pair and an infinite set of conservation laws for the Ernst equation is described. The matrix form of this equation is rewritten as a differential ideal of $gl(2,R)$-valued differential forms, and its symmetry condition is expressed as an exterior equation which is linear in the symmetry characteristic and has the form of a conservation law. By means of a recursive process, an infinite collection of such laws is then obtained, and the conserved "charges" are used to derive a linear exterior equation whose components constitute a Lax pair.

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1. Introduction

The search for the connections between symmetry and integrability has always been a central problem in the study of nonlinear partial differential equations (PDEs). For those PDEs having an underlying variational structure, the work of E. Noether and its extensions (see, e.g., [1,2]) provide an important link between variational symmetries and conservation laws. Non-variational connections between symmetry and integrability, however, also exist. They are often related to the possibility of "linearizing" a nonlinear PDE by use of a Lax pair, i.e., a pair of coupled PDEs linear in an auxiliary function ψ and integrable for ψ on the condition that the original (nonlinear) PDE is satisfied.Linearity is an important issue here,since the symmetry condition (characteristic equation) of a

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PDE is itself a linear PDE for the symmetry characteristic [1,2].

A given nonlinear PDE may often be linearized in more than one way by different choices of a Lax pair. A particularly useful choice is the one in which the Lax pair plays the role of a Bäcklund transformation connecting the PDE with its symmetry condition [3], so that the solution ψ of the pair is a symmetry characteristic for the PDE (or, more generally, is linearly dependent on a symmetry characteristic). Hence, in a sense, the symmetry condition is "built" into the Lax pair. In this way, one obtains a symmetry of the PDE by integrating the associated linear system.

A well-known example where these ideas find wide applications is the self-dual Yang-Mills equation [4,5]. Interestingly, this has been shown to be a sort of prototype equation from which several other known PDEs are derived by reduction [6,7]. One such PDE is the Ernst equation of General Relativity describing stationary, axially symmetric gravitational fields. In a previous paper [8] the authors proposed a new Lax pair for this equation (an older one was found by Belinski and Zakharov [9]) and showed that the solution ψ of this pair is indeed linearly related to a symmetry characteristic. In addition to giving new "hidden" symmetries, the Lax pair also leads to the construction of infinite collections of conservation laws for the Ernst equation.

Admittedly, finding a Lax pair with specific properties almost always requires a certain amount of guessing, as well as a lot of patience in a long trial-and-error process. We now ask the question: Can a linear system such as that of [8] be derived in a systematic way? This article answers this question in the affirmative. As we show, the symmetry condition alone leads one straightforwardly to the discovery of infinite sets of conservation laws, as well as a Lax pair having the desired properties. Our formalism is expressed in the language of exterior differential forms which is both elegant and economical. Hence, for example, differential equations expressing conservation laws, as well as systems of PDEs constituting differential recursion relations or Lax pairs, will now be represented by single exterior equations. In this regard, it would be more appropriate to speak of an *exterior linearization equation*, rather than of a Lax pair in the ordinary sense of this term.

In short, the process is as follows: First, we rewrite the Ernst equation as a differential ideal of matrix-valued differential forms and express its symmetry condition as an exterior equation which is linear in the symmetry characteristic. This latter equation is in conservation-law form, and this fact allows us to introduce a first "conserved charge" or "potential". A second conservation law is then found, with a new potential, and the process continues indefinitely, yielding a double infinity of conserved charges. These charges are related to each other via a certain recursion relation and are used as Laurent coefficients in a series whose terms involve powers (both positive and negative) of a complex "spectral" parameter. This series (assuming it converges) represents some complex function Ψ, which is shown to satisfy an exterior linearization equation equivalent to a Lax pair.

2. Mathematical Preliminaries

The variables $x^\mu \equiv \rho,\ z$ (μ=1,2, respectively) will be regarded as local orthogonal coordinates in a 2-dimensional Euclidean space with metric $\delta_{\mu\nu}$. Geometrical objects defined in this space (such as functions or differential forms) are assumed matrix-valued, with values generally in $gl(2,C)$ (with appropriate restrictions, such as real-valuedness, etc., in accordance with physical requirements).

The volume 2-form in our space is

$$\tau = \ ^1/_2\ \varepsilon_{\mu\nu}\, dx^\mu dx^\nu = \ d\rho\, dz$$

(the usual summation convention is assumed). For any 1-form

$$\sigma = \sigma_\mu\, dx^\mu = \ \sigma_1\, d\rho + \sigma_2\, dz,$$

the *dual* of σ with respect to τ is defined as the 1-form $^*\sigma$ with components

$$(*\sigma)_\nu = \ \tau_{\mu\nu}\sigma^\mu = \ \varepsilon_{\mu\nu}\,\delta^{\mu\lambda}\sigma_\lambda \quad ,$$

so that

$$*\sigma = \ (*\sigma)_\mu\, dx^\mu = \ -\sigma_2\, d\rho + \sigma_1\, dz.$$

In particular, $^*d\rho{=}dz$, $^*dz = -\,d\rho$. Also,

$$*(*\sigma) = -\,\sigma \tag{1}$$

For 1-forms σ_1 and σ_2, we have that

$$*\sigma_1 \wedge *\sigma_2 = \ \sigma_1 \wedge \sigma_2\ , \qquad \sigma_1 \wedge *\sigma_2 = -\,(*\sigma_1) \wedge \sigma_2 \tag{2}$$

We note that the * operation is linear, so that

$$*(\alpha\,\sigma_1 + \beta\,\sigma_2) = \ \alpha\ *\sigma_1 + \beta\ *\sigma_2 \tag{3}$$

where α and β are 0-forms.

Given any differential forms ζ and ξ, we define the commutator

$$[\zeta,\ \xi]\ \equiv\ \zeta \wedge \xi - \xi \wedge \zeta.$$

In particular, if σ is a 1-form and ψ is a 0-form, then $[\sigma{,}\psi]{=}\sigma\psi - \psi\sigma$ and, by the antiderivation property of the exterior derivative,

$$d\,[\sigma,\ \psi]\ =\ [d\sigma,\ \psi]\ -\ \{\sigma,\ d\psi\} \tag{4}$$

where, in general, curly brackets denote anticommutators:

$$\{\sigma_1\ ,\ \sigma_2\} \equiv\ \sigma_1 \wedge \sigma_2 + \sigma_2 \wedge \sigma_1.$$

We note that, to simplify our notation, we will often omit the symbol $\wedge$ of the exterior product. It should be kept in mind, however, that exterior multiplication of differential forms will always be assumed. Thus, an expression like $\sigma_1\sigma_2$ should be understood as $\sigma_1{\wedge}\sigma_2$.

3. Ernst Equation: Geometrical Formulation and Symmetry

We adopt the following matrix form of the Ernst equation [6,7]:

$$(\rho g^{-1} g_\rho)_\rho + (\rho g^{-1} g_z)_z = 0 \tag{5}$$

where subscripts denote partial derivatives with respect to the variables ρ, z, collectively denoted x^μ (μ=1,2, respectively). The matrix function g is assumed to be $SL(2,R)$-valued and symmetric. With the parametrization

$$g = \frac{1}{f} \begin{bmatrix} 1 & \omega \\ \omega & f^2 + \omega^2 \end{bmatrix}$$

and by setting $E = f + i\omega$, we recover the Ernst equation in the usual form,

$$(ReE)\, \nabla^2 E = (\nabla E)^2.$$

With the substitutions

$$A = g^{-1} g_\rho \ , \quad B = g^{-1} g_z \ ,$$

equation (5) becomes equivalent to the system of PDEs

$$A + \rho\,(A_\rho + B_z) = 0 \tag{6}$$

$$B_\rho - A_z + \ [A\,,\, B] = 0 \tag{7}$$

The second equation is just the integrability condition in order that g may be reconstructed from A and B.

We introduce the matrix-valued "connection" 1-form

$$\gamma = g^{-1} dg = \ A\, d\rho + B\, dz \tag{8}$$

The integrability condition $d(dg)$=0 in order that g may be recovered from γ, together with the obvious requirement that g be nonsingular, yield the Mauer-Cartan equation ω=0, where ω is the 2-form

$$\omega \ = d\gamma + \gamma \wedge \gamma = dB\, dz - d\rho\, dA + \ [A\,,\, B]\, d\rho dz \tag{9}$$

We also construct the 2-form

$$d\,(\rho * \gamma) = \ A\, d\rho\, dz + \ \rho\,(dA\, dz + d\rho\, dB) \tag{10}$$

where ${}^*\gamma$ =-$Bd\rho$+Adz.

We now observe that Eqs.(6) and (7) correspond to the system of exterior equations

$$d\,(\rho \ * \gamma) = 0 \ , \qquad \omega \ = 0 \tag{11}$$

Indeed, one may consider $d(\rho^*\gamma)$ and ω as 2-forms in a jet-like space of four variables: the scalar variables $x^\mu=\rho,z$ and the $gl(2,R)$ variables A and B. Equations (6) and (7) are recovered by projecting Eqs.(11) onto the base space of the x^μ.

Let $I\{d(\rho^*\gamma),\, \omega\}$ be the ideal of forms [10-12] generated by the 2-forms $d(\rho^*\gamma)$ and ω. The first form is exact, thus its exterior derivative is trivially a member of the ideal, while, as we can easily show, $d\omega = \omega\wedge\gamma - \gamma\wedge\omega$, which also belongs to I. We thus conclude that I is a differential (closed) ideal.

The first of Eqs.(11) implies the existence of a matrix potential X such that $\rho^*\gamma=dX$ (that is, $\rho A=X_z$, ρB=-X_ρ).Then, $^*dX$=- $\rho\gamma$, and, by the Mauer-Cartan equation ω=0, we get

$$d\rho * dX - \rho\, d * dX + dX\, dX = 0 \tag{12}$$

[where use has been made of the first of Eqs.(2)]. In component form,

$$X_\rho - \rho\,(X_{\rho\rho} + X_{zz}) + [X_\rho\,,\, X_z] = 0 \tag{13}$$

We introduce the covariant derivatives

$$D_\rho = \partial_\rho + [A\,,\ ]\ , \quad D_z = \partial_z + [B\,,\ ] \tag{14}$$

(where $\partial_\rho=\partial/\partial\rho$ and $\partial_z=\partial/\partial z$) which are seen to be derivations on the Lie algebra of $gl(2,C)$-valued functions. We also define an exterior covariant derivative D which acts on $gl(2,C)$ functions Φ as follows:

$$D\Phi = d\Phi + [\gamma,\, \Phi] = (D_\rho\Phi)\, d\rho + (D_z\Phi)\, dz \tag{15}$$

We now look at the symmetry problem for system (11). We first note that all symmetries of a system of PDEs can be expressed as infinitesimal transformations of the dependent variables alone [1,2]. Thus, all symmetries may be represented by “vertical” vector fields, i.e., vectors with vanishing projections on the base space of the x^μ. Let $\delta g=\alpha Q[g]$ be an infinitesimal symmetry transformation of Eq.(5), where α is an infinitesimal parameter and Q is a matrix-valued function which may depend locally or nonlocally on g. It is convenient to set $Q=g\Phi$, where Φ is another matrix 0-form. The infinitesimal symmetry of Eq.(5) is then written as

$$\delta g = \alpha\, g\, \Phi \tag{16}$$

(with appropriate restrictions on Φ in order that the transformation preserve the symmetric $SL(2,R)$ character of g). This induces the symmetry transformations $\delta\mathrm{A} = \alpha D_\rho\Phi$, $\delta B=\alpha D_z\Phi$ of system (6)-(7). These are summarized by the *formal* vector field

$$V = D_\rho\Phi\, \frac{\partial}{\partial A} + D_z\Phi\, \frac{\partial}{\partial B} \tag{17}$$

The symmetry condition on the ideal I of the 2-forms $d(\rho^*\gamma)$ and ω is that the Lie derivative with respect to V should leave this ideal invariant [10-12]:

$$L_V I \subset I \quad .$$

This is satisfied by requiring that

$$L_V\, d(\rho * \gamma) = L_V \omega \;=\; 0 \quad \mathrm{mod} \quad I\,\{d(\rho * \gamma),\, \omega\,\} \tag{18}$$

By using Eq.(9) for ω, taking into account that the Lie derivative commutes with the exterior derivative and satisfies the Leibniz rule, and by noting that

$$L_V \gamma = L_V\,(A\, d\rho + B\, dz) = (D_\rho \Phi)\, d\rho + (D_z \Phi)\, dz \;=\; D\Phi \;=\; d\Phi + [\gamma,\, \Phi] \quad ,$$

we find that

$$L_V \omega \;=\; \omega\, \Phi - \Phi\, \omega \;\equiv\; [\omega\,,\, \Phi],$$

which is automatically a member of the ideal I, hence satisfies the condition for ω in Eq.(18). On the other hand, by noting that

$$L_V * \gamma = L_V(-B\, d\rho + A\, dz) = \; *\, D\, \Phi,$$

we find that the condition for $d(\rho\, {}^*\gamma)$ is expressed as an exterior equation which is linear in Φ:

$$d\,(\rho * D\,\Phi) = 0 \quad \textit{on solutions} \tag{19}$$

(where "on solutions" means: when Eqs.(11) are satisfied). In component form,

$$(\rho D_\rho \Phi)_\rho \;+\; (\rho D_z \Phi)_z = 0 \quad \textit{on solutions} \tag{20}$$

The reader is invited to derive the symmetry condition (20) directly from the Ernst equation (5) by assuming a symmetry characteristic $Q{=}g\Phi$ and by applying the abstract formalism described in [3]. (Note, however, that our present notation is different from that of [3]. Specifically, the symbols D_ρ and D_z, which here denote covariant derivatives, have the meaning of *total* derivatives in [3].)

4. Conservation Laws and Exterior Linearization Equation

We now turn to integrability characteristics of the Ernst equation. As is well known, the hallmark of integrability is the existence of a linear system or Lax pair. This system may be compactified into a single exterior equation involving 1-forms, which will be referred to as an *exterior linearization equation.* The purpose of this section is to describe a systematic construction of such a linearization equation for the Ernst equation, or equivalently, for the exterior system (11).

We begin with the symmetry condition (19):

$$d\,(\rho * D\,\Phi) = 0 \tag{21}$$

The corresponding infinitesimal symmetry transformation is $g' = g + \alpha g \Phi$, according to Eq.(16). This means that g' is a solution of the general PDE (5) when g is a solution. However, we will not require here that the new solution g' conform to the extra physical restrictions imposed on the original solution g, namely, of being symmetric and having unit determinant. Thus, all real solutions Φ of the exterior equation (21) will be admissible (e.g., $\Phi = g^{-1}g_z = B$).

As its component form (20) suggests, the exterior equation (21) expresses a conservation law valid for solutions of the Ernst equation. Equation (21) also implies the existence of a "conserved charge" or "potential" Φ', such that

$$d\Phi' = \rho * D\,\Phi \; = \rho\,(*d\Phi + [*\gamma,\, \Phi]\,)$$

[where use has been made of the linearity property (3) of the star operation]. Starring this equation, solving for $d\Phi$, and requiring that $d(d\Phi)$=0, we find another conservation law:

$$d\,(\rho * D\,\Phi' - 2\Phi' dz) = \; 0 \quad ,$$

by which we introduce a new potential Φ'' such that

$$d\Phi'' \; = \rho * D\,\Phi' - 2\Phi' dz \; = \rho\,(*d\Phi' + [*\gamma,\, \Phi']\,) - 2\Phi' dz \quad .$$

Starring this and applying $d(d\Phi')$=0, we obtain yet another conservation law:
$d\,(\rho * D\,\Phi'' \; - 4\,\Phi''\,dz) = \; 0$, etc.

This process suggests that we consider the following exterior recursion relation:

$$\begin{aligned} d\Phi^{(n+1)} \; &= \rho * D\,\Phi^{(n)} - 2n\,\Phi^{(n)}dz \\ &= \; \rho\,(*d\Phi^{(n)} + \; [*\gamma,\, \Phi^{(n)}]\,) - 2n\,\Phi^{(n)}dz \end{aligned} \tag{22}$$

with $\Phi^{(0)}$=Φ representing a symmetry characteristic of the Ernst equation in its general form (5) [i.e., a solution of Eq.(21)].

In order that the exterior equation (22) be integrable for $\Phi^{(n+1)}$ for an already known $\Phi^{(n)}$, the integrability condition $d(d\Phi^{(n+1)})$=0 must be satisfied. This yields

$$d\;\left(\rho * D\,\Phi^{(n)} - 2n\,\Phi^{(n)}dz\right) = \; 0 \tag{23}$$

We will now show that Eq.(23) is a conservation law valid for solutions of the Ernst equation. The left-hand side of (23) is written as

$$\begin{aligned} l.h.s.\,(23) = \; & d\,(\rho \; * d\Phi^{(n)} + \; [\rho * \gamma,\, \Phi^{(n)}] - 2n\,\Phi^{(n)}dz) \\ = \; & d\rho \; * d\Phi^{(n)} + \rho\, d * d\Phi^{(n)} + \; d\,[\rho * \gamma,\, \Phi^{(n)}] - 2n\, d\Phi^{(n)}dz \;. \end{aligned}$$

By using property (4) and the second property (2), we have:

$$\begin{aligned} d\,[\rho * \gamma,\, \Phi^{(n)}] &= [d(\rho * \gamma),\, \Phi^{(n)}] \; - \; \rho * \gamma\, d\Phi^{(n)} - \; \rho\, d\Phi^{(n)} \; * \gamma \\ &= \; [d(\rho * \gamma),\, \Phi^{(n)}] \; + \; \rho\,\gamma \; * d\Phi^{(n)} + \; \rho \; * d\Phi^{(n)}\gamma \;, \\ d\Phi^{(n)}dz &= \; d\Phi^{(n)} \; * d\rho = \; d\rho \; * d\Phi^{(n)} \,. \end{aligned}$$

Therefore,

$$l.h.s.\,(23) = \;(1-2n)\,d\rho \ast d\Phi^{(n)} + \rho\, d\ast d\Phi^{(n)} + \;[d(\rho\ast\gamma),\,\Phi^{(n)}] + \rho\,\gamma\ast d\Phi^{(n)} + \rho \ast d\Phi^{(n)}\gamma.$$

Now, by rewriting the recursion relation (22) with $(n-1)$ in place of n, we can express $d\Phi^{(n)}$, thus also ${}^*d\Phi^{(n)}$, in terms of $\Phi^{(n-1)}$. Substituting for ${}^*d\Phi^{(n)}$ into the expression for the l.h.s. of (23), and taking into account that $d\gamma + \gamma\gamma = \omega$, we finally find:

$$l.h.s.\,(23) = \;[d\,(\rho\ast\gamma)\,,\;\Phi^{(n)}] \;-\; \rho^2\,[\omega\,,\;\Phi^{(n-1)}] \quad .$$

We note that this expression vanishes when $d(\rho\,{}^*\gamma)$=0 and ω=0, i.e., for solutions of the Ernst equation. This proves the conservation-law property of Eq.(23).

As we have just shown, the conservation law (23) is the necessary condition for $\Phi^{(n)}$ in order that the exterior equation (22) be integrable for $\Phi^{(n+1)}$. For n =0, Eq.(23) is just the symmetry condition (21), which is indeed satisfied by $\Phi^{(0)}$ since the latter is, by assumption, a symmetry characteristic. Now, we must show that the solution $\Phi^{(n+1)}$ of Eq.(22) also conforms to condition (23) with $(n+1)$ in place of n. This will ensure that the recursive process may continue indefinitely for all values of n, yielding an infinite number of conservation laws from any given symmetry characteristic $\Phi^{(0)}$. This time we need to eliminate $\Phi^{(n)}$ from Eq.(22) in favor of $\Phi^{(n+1)}$. By this process we will actually derive the necessary condition for $\Phi^{(n+1)}$ in order that the exterior equation (22) be integrable for $\Phi^{(n)}$ when $\Phi^{(n+1)}$ is already known. This will allow us to use the recursion relation (22) "backwards" to obtain potentials $\Phi^{(n)}$ and corresponding conservation laws (23) for *negative* values of n also. Thus, the validity of Eqs.(22) and (23) will be extended to all integral values n =0, ±1, ±2, ...

Starring Eq.(22) and solving for $d\Phi^{(n)}$, we get:

$$d\Phi^{(n)} = \;-\;\frac{1}{\rho}\;\ast\,d\Phi^{(n+1)} \;-\; [\,\gamma,\;\Phi^{(n)}] \;+\; \frac{2n}{\rho}\,\Phi^{(n)}d\rho \tag{24}$$

We apply the integrability condition $d(d\Phi^{(n)})$=0, and use Eq.(24) again to replace $d\Phi^{(n)}$ where it appears. Then, a lengthy but relatively straightforward calculation, performed with the aid of properties (2) and (4), shows that

$$d\,\left(\rho\ast D\,\Phi^{(n+1)} - 2\,(n+1)\,\Phi^{(n+1)}dz\right) = \;[d\,(\rho\ast\gamma)\,,\;\Phi^{(n+1)}] \;-\; \rho^2\,[\omega\,,\;\Phi^{(n)}] \quad .$$

So, the left-hand side of the above equation vanishes for solutions of the Ernst equation, as it should.

In conclusion, starting with any symmetry characteristic $\Phi^{(0)}$, we can use the recursion relation (22) to find a double infinity of conserved charges (potentials) $\Phi^{(n)}$ for $n = \pm1$, ±2, ... These charges are increasingly nonlocal in g, since they involve integrals of increasing order of expressions containing the function g.

With these charges in hand, we now introduce a complex variable λ (to be identified with a spectral parameter) and construct a function $\Psi(x^\mu,\lambda)$ having the following series representation for $\lambda\neq0$:

$$\Psi(x^\mu,\,\lambda) = \sum_{n=-\infty}^{+\infty} \lambda^n\,\Phi^{(n)}(x^\mu) \tag{25}$$

We assume that the series (25) converges to the function Ψ which is single-valued and analytic (as a function of λ) in some annular region centered at the origin of the λ-plane. Hence, Eq.(25) represents a Laurent expansion of Ψ in this region.

Multiplying the recursion relation (22) by λ^n, summing over all integral values of n, and using Eq.(25), we find an exterior equation linear in Ψ:

$$\rho\, *\, D\Psi - 2\lambda\Psi_\lambda\, dz \;=\; \frac{1}{\lambda}\, d\Psi \tag{26}$$

or explicitly,

$$\rho\, *\, d\Psi + \;[\rho * \gamma,\; \Psi] - 2\lambda\Psi_\lambda\, dz \;=\; \frac{1}{\lambda}\, d\Psi \tag{27}$$

Relation (26) is an *exterior linearization equation* for the Ernst equation, equivalent to a Lax pair. Specifically, the exterior equation (26), linear with respect to Ψ, is integrable for Ψ when the exterior equations (11) are satisfied.

The proof of this statement is outlined as follows: The integrability condition for solution of Eq.(26) is $d(d\Psi)$=0. So, the exterior derivative of the left-hand side of this equation must vanish. By using algebraic manipulations which are by now familiar to the reader (such as, for example, $\{^*\gamma, d\Psi\}=-\{\gamma, {}^*d\Psi\}$, $d\Psi_\lambda dz = d\rho^* d\Psi_\lambda$, etc.), the above requirement leads to the following exterior equation:

$$d\rho\, *\, d\,\Psi + \rho\, d\, *\, d\,\Psi + \;[d(\rho * \gamma)\,,\; \Psi] \;+\; \rho\{\,\gamma,\; *\, d\,\Psi\} - 2\lambda\, d\rho\, *\, d\,\Psi_\lambda \;=\; 0 \tag{28}$$

By starring the linear system (27), we find an expression for $^*d\Psi$:

$$*d\,\Psi = \; -\lambda\rho\,(d\,\Psi + \;[\gamma,\,\Psi]\,) + 2\lambda^2\,\Psi_\lambda\, d\rho \tag{29}$$

Differentiating this with respect to λ, we have:

$$*d\,\Psi_\lambda = \; -\rho\,(d\,\Psi + \;[\gamma,\,\Psi]\,) \; -\lambda\rho\,(d\,\Psi_\lambda + \;[\gamma,\,\Psi_\lambda]\,) + 4\lambda\Psi_\lambda\, d\rho + 2\lambda^2\,\Psi_{\lambda\lambda}\, d\rho$$

Substituting this equation and Eqs.(29) into the integrability condition (28), we finally get:

$$[\, d\,(\rho * \gamma) - \lambda\rho^2\omega\;,\;\; \Psi\,] \;=\; 0 \quad ,$$

where $\omega = d\gamma + \gamma\gamma$. The above relation is valid independently of Ψ and λ if $d(\rho^*\gamma)$=0 and ω=0, i.e., for solutions of the Ernst equation. This proves that the integrability of the exterior equation (26) for Ψ is indeed dependent upon the satisfaction of the Ernst equation.

In component form, Eq.(26) is written as a pair of linear first-order PDEs for Ψ:

$$\begin{aligned} \rho D_\rho\,\Psi - 2\lambda\Psi_\lambda \;&=\; \tfrac{1}{\lambda}\;\Psi_z \\ \rho D_z\,\Psi \;\;&=\;\; -\;\tfrac{1}{\lambda}\;\Psi_\rho \end{aligned} \tag{30}$$

The reader is invited to show that the integrability of system (30) for Ψ requires that equation (5) is satisfied (see also [8]). Thus, (30) represents a Lax pair for the Ernst equation. In fact, this pair is equivalent to that found by different means in [8]. What we have shown is that this system may actually be constructed by a remarkably straightforward process, by starting with the symmetry condition of the field equation.

5. Connection to Other Linear Systems

It can be shown (see [8,3]) that, by solving the linear system (30) for Ψ, for a given solution g of the Ernst equation, one simultaneously obtains an infinitesimal "hidden" symmetry of this equation, given by the expression

$$\delta g = \frac{\alpha}{2\pi i} \int_C \frac{d\lambda}{\lambda} \left(g\,\Psi(x^\mu, \lambda) + \Psi^T(x^\mu, \lambda)\, g \right) \tag{31}$$

where α is an infinitesimal parameter, C is a positively oriented, closed contour around the origin of the λ-plane, and Ψ^T denotes the transpose of the matrix Ψ. (Here, g is assumed to conform to the physical restrictions of being real, symmetric, and of unit determinant. Moreover, Ψ is required to be traceless and to assume real values when λ is confined to the real axis. Then, the new solution $g' = g + \delta g$ obeys the same physical restrictions as g.) Since solutions of the system (30) [or equivalently, the exterior linearization equation (26)] are of importance in this regard, any mechanism for producing as many solutions as possible would be useful. We now exhibit a simple transformation which maps solutions of (a form of) the Belinski-Zakharov (B-Z) linear system [9] into solutions of our linearization equation (26).

We recall the exterior linearization equation (27):

$$\rho\,(*d\Psi + [*\gamma,\ \Psi]\,) - 2\lambda \Psi_\lambda\, dz \ = \ \frac{1}{\lambda}\, d\Psi \tag{32}$$

where Ψ conforms to the physical conditions mentioned in the previous paragraph; namely, $tr\Psi$=0 and $\Psi(x^\mu,\lambda^*)$=$\Psi^*(x^\mu,\lambda)$ (the asterisk here denotes complex conjugation). On the other hand, a variant form of the B-Z linear system, adapted to the particular form of our equations, is the following:

$$\rho\,(*d\Phi + *\gamma\,\Phi) - 2\lambda \Phi_\lambda\, dz \ = \ \frac{1}{\lambda}\, d\Phi \tag{33}$$

Let $\Phi(g;\lambda)$ be a non-singular solution of the exterior equation (33) for some solution g of the Ernst equation. We assume that Φ becomes real for real values of λ. Consider now the function $\Psi(g;\lambda)$ given by

$$\Psi = \Phi\, T\, \Phi^{-1} \tag{34}$$

where T is an arbitrary traceless matrix function of the form

$$T = F\, \left(z - \frac{\lambda \rho^2}{2} + \frac{1}{2\lambda} \right) \tag{35}$$

subject to the condition that F be real-valued for real values of λ. It may then be proven that $\Psi(g;\lambda)$ is a solution of the linearization equation (32).

Although only a subset of the entirety of solutions of Eq.(32) can be produced in this fashion, the transformation (34)-(35) is an effective way of taking advantage of our knowledge regarding the B-Z formulation for the purpose of finding hidden symmetries of the Ernst equation.

Our method for finding a linear system and an infinite number of nonlocal conserved currents for the Ernst equation is closely related to that of Nakamura [13]. In the latter case, the Lax pair does not contain derivative terms with respect to the spectral parameter. Moreover, the infinite set of conservation laws is accompanied by a corresponding infinite set of nonlocal symmetries, which is not the case with our method for the Ernst equation but which *is* the case with regard to another familiar nonlinear system, the self-dual Yang-Mills (SDYM) equation. To achieve these extra characteristics, however, one has to perform an analytic continuation of $g(\rho,z)$ into complex space and introduce more independent variables. In this way the Ernst equation transforms into a reduced form of the SDYM equation, and the mathematical treatments of these two systems become quite similar.

Summary

In this article we have pursued our study of the relation between symmetry and integrability characteristics of the Ernst equation. Taking advantage of the conservation-law form of the symmetry condition, we have inductively produced a double infinity of nonlocal conserved charges by means of a recursion relation. These charges were then used as Laurent coefficients in an infinite series whose terms involve powers (both positive and negative) of a complex "spectral" parameter. Within its domain of convergence, this series represents a function Ψ which is seen to satisfy a certain linear system, the integrability of which for Ψ is possible in view of the Ernst equation. Finally, we have presented a simple transformation which maps all solutions of the Belinski-Zakharov Lax pair [9] into solutions of our linear system, and we have compared our results to those of Nakamura [13]. Our formalism was developed in the language of differential forms and exterior calculus, which allowed us to present our equations in a more compact, as well as a more elegant form.

It is remarkable that integrability properties of the Ernst equation, such as the existence of Lax pairs and an infinite number of conservation laws, can be derived in a straightforward way by performing rather natural manipulations on the symmetry condition. This characteristic, which is also observed in the case of the SDYM equation, reveals a profound, non-Noetherian connection between symmetry and integrability. It will be further explored in future publications.

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Symmetry, Conserved Charges, and Lax Representations of Nonlinear Field Equations: A Unified Approach

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Abstract: A certain non-Noetherian connection between symmetry and integrability properties of nonlinear field equations in conservation-law form is studied. It is shown that the symmetry condition alone may lead, in a rather straightforward way, to the construction of a Lax pair, a doubly infinite set of (generally nonlocal) conservation laws, and a recursion operator for symmetries. Applications include the chiral field equation and the self-dual Yang-Mills equation.

Keywords: Nonlinear Equations; Symmetries; Conservation Laws; Lax Pair
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1. Introduction

In a recent paper [1] an analytical method was described for constructing a Lax pair for the Ernst equation of General Relativity. The starting point was the symmetry condition (or linearized form) of the field equation. The latter equation is in conservation-law form, and thus so is its associated symmetry condition. A doubly infinite hierarchy of conservation laws was then constructed by a recursive process, and the conserved "charges" were used as Laurent coefficients in a series representation (in powers of the spectral parameter) of a function Ψ which was seen to satisfy the sought-for Lax pair.

It is natural to inquire whether this technique can also be applied to other nonlinear partial differential equations (PDEs) of Mathematical Physics. This article describes a general, non-Noetherian framework for connecting integrability characteristics of a given nonlinear PDE to the symmetry properties of this PDE. It is remarkable that, by starting

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with the symmetry condition, one may discover a number of important things such as the existence of a recursion operator [2,3] for symmetries, a doubly-infinite set of (typically nonlocal) conservation laws, and a Lax pair which "linearizes" the nonlinear field equation.
To illustrate the use of the method, application is made to two familiar nonlinear PDEs: the chiral field equation and the self-dual Yang-Mills equation. In these examples, the corresponding Lax pairs and infinite sequences of conservation laws are constructed explicitly. Moreover, the recursion operators for symmetries are derived. In the case of the real Ernst equation, treated previously in [1], although a recursion operator doesn't seem to exist for that particular form of the equation (due to the coordinate "pathology" which results in the explicit appearance of an independent variable in the PDE), one still gets an interesting "hidden" symmetry transformation which leads to new approximate solutions for stationary gravitational fields with axial symmetry [4].

2. The General Idea

Let $F\ [u]=0$ be a nonlinear PDE in the dependent variable u and the independent variables $x,\ y,\ \ldots$. The bracket notation $[u]$ indicates that the function F may depend explicitly on the variables $u,\ x,\ y,\ \ldots$, as well as on partial derivatives, of various orders, of u with respect to the independent variables, denoted $u_x, u_y, u_{xx}, u_{yy}, u_{xy}$, etc. We adopt the definition according to which the PDE is integrable if it has an associated Lax-pair representation, i.e., if it can be expressed as an integrability condition for solution of a linear system of PDEs for an auxiliary field Ψ :

$$L_i\,(\Psi\,;\ u\,;\ \lambda)\ =\ 0 \qquad i=1,2 \tag{1}$$

where the differential expressions L_i are linear in Ψ, and where λ is a (generally complex) "spectral" parameter.
It has been observed that integrable PDEs often have an infinite number of symmetries which may be produced, for example, with the aid of one or more recursion operators (see, e.g., [2,3] and the references therein). This connection between symmetry and integrability may be attributed to a variety of factors. For example, an integrable PDE may have an underlying Hamiltonian structure in which the Lagrangian density possesses an infinite number of variational symmetries. In this case, the Noether theorem provides the connection between symmetry and integrability, the latter manifesting itself in the presence of an infinite set of conservation laws. As is often the case, the existence of these laws is associated with a Lax structure for the nonlinear problem.
Non-Noetherian connections between symmetry and integrability, however, may also exist. Let us recall that the nonlinear PDE $F\ [u]=0$ is a consistency condition for solution of the system (1). On the other hand, the (generally complex) function Ψ will satisfy some PDE of its own, also derived from system (1). This PDE will be linear in Ψ and will contain u as a "parametric" function:

$$G\,(\Psi\,;\,u)\;=\;0 \tag{2}$$

where the expression G is linear in Ψ, and where u is a solution of $F\,[u]{=}0$. We may say that the system (1) is a Bäcklund transformation relating the nonlinear PDE $F\,[u]{=}0$ to the linear PDE (2). Now, we already know an equation of the form (2): it is the *symmetry condition* (linearized form) of $F\,[u]{=}0$. Let $u' = u{+}\alpha\,Q\,[u]$ be an infinitesimal symmetry transformation for the latter PDE, where α is an infinitesimal parameter (we note that any symmetry of a PDE can be expressed as a transformation of the dependent variable alone [2,3], i.e., is equivalent to a "vertical" symmetry). The *symmetry characteristic* Q $[u]$ then satisfies a linear PDE of the form

$$S\,(Q\,;\,u)\;=\;0\;\;mod\,F\,[u] \tag{3}$$

where "mod $F\,[u]$" signifies that the PDE on the left is satisfied when u is a solution of the nonlinear PDE $F\,[u]{=}0$. Now, if it happens that Eqs.(2) and (3) become identical when $\Psi{\equiv}Q$ (i.e., if the functions G and S are the same), then the solution Ψ of the Lax pair (1) will also be a symmetry characteristic of $F\,[u]{=}0$:

$$S\,(\Psi\,;\,u)\;=\;0\;\;mod\,F\,[u] \tag{4}$$

Of course, as the examples of the self-dual Yang-Mills equation [5] and the Ernst equation [1,4] have taught us, it is possible that a given nonlinear PDE admit more than one Lax representation. What we are seeking here is a Lax pair which functions as a Bäcklund transformation connecting the nonlinear PDE $F\,[u]{=}0$ to its (linear) symmetry condition (3). The symmetry condition itself is thus "built" into the Lax pair, and a very fundamental connection between symmetry and integrability is established.
With regard to the complex parameter λ of the Lax pair (1), we remark the following: Since the role of such a parameter is generally nontrivial, it will be required that λ be nonzero (as well as, of course, finite in magnitude). We then expect that the solution Ψ of system (1), for a given u satisfying $F\,[u]{=}0$, will be an analytic function of λ for $\lambda \neq 0$. This solution may thus be represented as a Laurent series expansion in powers of λ, with u-dependent coefficients:

$$\Psi(u\,;\,\lambda) = \sum_{n=-\infty}^{+\infty} \lambda^n\,Q^{(n)}[u] \tag{5}$$

where the functions $Q^{(n)}[u]$ may be local or nonlocal in u. Now, we recall that Ψ is assumed to be a symmetry characteristic of $F\,[u]{=}0$, and this must be true for all values of λ in the Lax pair. Substituting Eq.(5) into Eq.(4) (which is linear in Ψ), and equating coefficients of all λ^n to zero, we find a doubly infinite set of linear PDEs for $Q^{(n)}$, of the form

$$S\,\left(Q^{(n)}\,;\,u\right)\;=\;0\;\;mod\,F\,[u]\;\;,\quad n=0,\;\pm 1,\;\pm 2,\cdots \tag{6}$$

All Laurent coefficients $Q^{(n)}[u]$ are thus seen to be symmetry characteristics for the nonlinear PDE $F\,[u]{=}0$, and the presence of this infinite set of symmetries is intimately related to the Lax pair.
Substituting the expansion (5) into the Lax pair (1), and equating coefficients of all powers of λ to zero, we obtain a pair of linear PDEs containing $Q^{(n)}$ and (say) $Q^{(n+1)}$. In essence, this is a Bäcklund transformation for the symmetry condition (3). This differential recursion relation constitutes a *recursion operator* [2,3] for the PDE $F\,[u]{=}0$, in the spirit of a new perception of this concept originally proposed by this author [5,6] and, independently, by Marvan [7]. We thus have a method for the explicit construction of such an operator. Starting with any symmetry $Q^{(0)}$, we can, in principle, use this operator to derive a double infinity of symmetries $Q^{(n)}$ (although not all of them will necessarily be nontrivial).
Finally, suppose that $F\,[u]$ is a divergence, so that the PDE $F\,[u]{=}0$ has the form of a conservation law. Then, its symmetry condition (3) also is in such form. Given that an infinite number of symmetry characteristics $Q^{(n)}[u]$ are available, we immediately obtain a doubly infinite collection of conservation laws for $F\,[u]{=}0$ from Eq.(6) (where now the function S is a divergence). Typically, the recursion operator connecting the $Q^{(n)}$ to each other is an integro-differential operator; thus, the conserved "currents" are generally expected to be nonlocal in u .

3. Analytical Description of the Method

Our objective is the following: Given a nonlinear PDE $F\,[u]{=}0$ in conservation-law form, we seek a Lax pair whose solution is a symmetry characteristic for this PDE, and, in the process, we expect to derive a recursion operator for symmetries as well as an infinite set of (nonlocal) conservation laws. Although the solution u of the PDE may depend on more than two independent variables, we restrict ourselves to the case where $F\,[u]$ is a divergence in only two of them:

$$F\,[u] \;\equiv\; D_x A\,[u] \,+\, D_y B\,[u] \;=\; 0 \tag{7}$$

where D_x and D_y denote total derivatives (see Appendix), which will also be indicated by using subscripts: $D_x A \equiv A_x$, etc. We will assume, in general, that u is square-matrix-valued, and so are the functions A, B, F.
Let $\delta u = \alpha\, Q\,[u]$ be an infinitesimal symmetry of Eq.(7) (where α is an infinitesimal parameter and Q is the matrix-valued symmetry characteristic). We write, in finite form,

$$\Delta\, u \;=\; Q\,[u] \tag{8}$$

where, in general, Δ denotes the Fréchet derivative of any function $f\,[u]$, with respect to the characteristic Q (see Appendix). The symmetry condition for the PDE (7) is

$$\Delta\, F\,[u] \;=\; 0 \;\; mod\, F\,[u] \tag{9}$$

where

$$\Delta F[u] \;=\; D_x \Delta A[u] \;+\; D_y \Delta B[u]$$

(since Fréchet derivatives and total derivatives commute). Putting

$$\Delta A[u] \;\equiv\; G(Q\,;\,u)\;, \quad \Delta B[u] \;\equiv\; H(Q\,;\,u) \tag{10}$$

(where the functions G and H are linear in Q), we rewrite Eq.(9) in the form of a linear PDE for Q :

$$S(Q\,;\,u) \;\equiv\; D_x G(Q\,;\,u) \;+\; D_y H(Q\,;\,u) \;=\; 0 \;\; mod\, F[u] \tag{11}$$

We note that $S(Q\,;\,u)$ is a divergence, so that the symmetry condition (11) is a conservation law for the corresponding nonlinear PDE (7).
Equation (11) suggests that we introduce a "potential" function K , such that $G = K_y$ and $H = -K_x$ (subscripts denote total differentiations). We assume that K is linearly dependent on some new function Q', and we write:

$$G(Q\,;\,u) \;=\; D_y K(Q'\,;\,u) \;\;, \quad H(Q\,;\,u) \;=\; -\,D_x K(Q'\,;\,u) \tag{12}$$

Clearly, this system is integrable for Q' (mod $F\,[u]$) if Q satisfies the symmetry condition (11). The integrability requirement for Q , on the other hand, will yield some linear PDE for Q'. It is possible that, by an appropriate choice of the function $K(Q';u)$, this PDE will be just the symmetry condition (11) for Q':

$$S(Q'\,;\,u) \;=\; 0 \;\; mod\, F[u].$$

That is, Q' will also be a symmetry characteristic. The system (12) then constitutes a Bäcklund transformation (BT) for the symmetry condition (11). This BT may be viewed as an invertible recursion operator for symmetries of the nonlinear PDE (7). Such an operator will, in principle, produce a doubly infinite sequence of symmetry characteristics $Q^{(n)}$ (n= ±1, ±2, ...) from any given characteristic $Q^{(0)}$.
To better display the recursive character of the BT (12), we rewrite this system as follows:

$$\begin{aligned} G\left(Q^{(n)};\,u\right) \;&=\; D_y K\left(Q^{(n+1)};\,u\right) \\ H\left(Q^{(n)};\,u\right) \;&=\; -\,D_x K\left(Q^{(n+1)};\,u\right) \end{aligned} \tag{13}$$

(n= 0, ±1, ±2, ...), where G and H are linear in $Q^{(n)}$, while K is linear in $Q^{(n+1)}$. Now, since all the $Q^{(n)}$ satisfy the PDE (11), the BT (13) also yields a double infinity of conservation laws for the field equation (7):

$$D_x G\left(Q^{(n)};\,u\right) \;+\; D_y H\left(Q^{(n)};\,u\right) \;=\; 0 \;\; mod\, F[u] \tag{14}$$

Starting with a known symmetry characteristic Q , we can evaluate the conserved "charges" $Q^{(n)}$ (n= 0, ±1, ±2, ...) as follows: (a) We take the BT (13) with n=0 and set $Q^{(0)}$=Q on the left-hand side. Then, $Q^{(1)}$ is found by integration. To find $Q^{(2)}$ we similarly integrate the BT (13) with n=1, etc. We thus obtain all positively-indexed charges $Q^{(n)}$. (b) We take the BT (13) with $n= -1$ and set $Q^{(0)}=Q$ on the right-hand side. We then solve for $Q^{(-1)}$. Working similarly for $n= -2, -3, ...$, we obtain all negatively-indexed charges $Q^{(n)}$.

We now introduce a complex parameter λ, which we require to be nonzero and of finite magnitude. Multiplying both sides of Eq.(13) by λ^n, summing over all integral values of n, and taking into account that the functions G, H and K are linear in their respective Q's, we find the following pair of PDEs:

$$\begin{aligned} G\left(\sum_{n=-\infty}^{+\infty} \lambda^n Q^{(n)};\ u\right) &= \tfrac{1}{\lambda}\, D_y K\left(\sum_{n=-\infty}^{+\infty} \lambda^n Q^{(n)};\ u\right) \\ H\left(\sum_{n=-\infty}^{+\infty} \lambda^n Q^{(n)};\ u\right) &= -\tfrac{1}{\lambda}\, D_x K\left(\sum_{n=-\infty}^{+\infty} \lambda^n Q^{(n)};\ u\right) \end{aligned} \tag{15}$$

We set

$$\Psi(u\,;\,\lambda) = \sum_{n=-\infty}^{+\infty} \lambda^n Q^{(n)}[u] \tag{16}$$

Equation (16) has the form of a Laurent expansion of a complex function Ψ in powers of λ, for a given solution u of the field equation (7). We note that Ψ is a linear combination of symmetry characteristics of Eq.(7), hence Ψ itself is a symmetry characteristic of that PDE. Substituting Eq.(16) into Eq.(15), we rewrite the latter in the form of a system of linear PDEs for Ψ:

$$D_y K(\Psi\,;\,u) \;=\; \lambda\, G\,(\Psi\,;\,u)\ , \qquad D_x K(\Psi\,;\,u) \;=\; -\,\lambda\, H\,(\Psi\,;\,u) \tag{17}$$

The consistency of this system requires that Ψ satisfy the linear PDE (11),

$$S\,(\Psi\,;\,u) \;\equiv\; D_x G\,(\Psi\,;\,u) \;+\; D_y H\,(\Psi\,;\,u) \;=\; 0 \quad (mod\, F\,[u]).$$

This verifies that Ψ is a symmetry characteristic. Moreover, the system (17) is linear in Ψ, and its solvability demands that u satisfy the nonlinear PDE (7) [this was required from the start in order that the BT (13), by which the charges $Q^{(n)}$ appearing in the Laurent expansion (16) are defined, may be integrable for $Q^{(n)}$ and $Q^{(n+1)}$] . We thus conclude that the linear system (17) constitutes a Lax pair for the field equation (7), and that, moreover, the solution Ψ of this system is a symmetry characteristic of that equation.

A final comment before closing this section: The whole idea was based on the assumption that an auto-Bäcklund transformation of the form (12) exists for the symmetry condition (11). It is possible, however, that no choice for the function $K(Q'; u)$ in Eq.(12) exists such that Q' be a symmetry characteristic when Q is such a characteristic. In this case, the method described above may still furnish an infinite number of conservation laws as well as a Lax pair, albeit not a recursion operator for producing infinite sets of symmetries. Moreover, the solution Ψ of the Lax pair will no longer represent a symmetry of the field equation (although, of course, it will somehow be related to a symmetry, since the symmetry condition was the starting point for constructing the Lax pair). The example of the Ernst equation, examined in detail in [1], made this point clear. In this case, the absence of an infinite set of symmetries is not a property of the gravitational field equations themselves (which, when properly formulated, do exhibit such an infinite set [8]) but is a consequence of the chosen real form of the Ernst equation, in which a spatial coordinate makes an explicit appearance.

4. Chiral Field Equation

The chiral field equation (a two-dimensional reduction of the self-dual Yang-Mills equation, to be discussed later) is of the form

$$F\,[g] \;\equiv\; (g^{-1}g_t)_t + \;(g^{-1}g_x)_x \;=\; 0 \tag{18}$$

where g is a $GL(N,C)$-valued function of t and x (as usual, subscripts denote total differentiations with respect to these variables). Let $\delta g \;=\; \alpha\, Q\,[g]$ be an infinitesimal symmetry of Eq.(18), with symmetry characteristic $Q\;[g]$. We have that $\Delta g \;=\; Q\,[g]$, where Δ denotes the Fréchet derivative with respect to Q (see Appendix). Moreover, by the commutativity of the Fréchet derivative with total derivatives,

$$\begin{aligned}\Delta F\,[g] \;&=\; D_t \Delta(g^{-1}g_t) \;+ D_x \Delta(g^{-1}g_x)\\ &=\; D_t \hat{A}_t(g^{-1}Q) \;+\; D_x \hat{A}_x(g^{-1}Q)\end{aligned}$$

where we have introduced the "covariant derivative" operators

$$\hat{A}_t = \; D_t + \;[g^{-1}g_t\,, \quad]\;, \qquad \hat{A}_x = \; D_x + \;[g^{-1}g_x\,, \quad]$$

(the square brackets denote commutators). It can be shown that these operators commute, as expected from the fact that the "connections" $g^{-1}g_t$ and $g^{-1}g_x$ are pure gauges. The symmetry condition (9) reads:

$$S(Q\,;\,g) \;\equiv\; (D_t \hat{A}_t \,+\, D_x \hat{A}_x)\,(g^{-1}Q) \;=\; 0 \;\; mod\, F\,[g] \tag{19}$$

and it is obviously in conservation-law form.

We now seek an auto-Bäcklund transformation (BT) of the form (12) for the linear PDE (19). This must be of the form

$$\hat{A}_t(g^{-1}Q) = \; K_x \;\;, \qquad \hat{A}_x(g^{-1}Q) = \; -\, K_t$$

for some function $K(Q';g)$. Let us try $K(Q';g) = g^{-1}Q'$:

$$\hat{A}_t(g^{-1}Q) = \;(g^{-1}Q'\,)_x \;\;, \qquad \hat{A}_x(g^{-1}Q) = \; -\,(g^{-1}Q'\,)_t \tag{20}$$

Integrability for Q' clearly requires that Q satisfy Eq.(19). The integrability condition for Q can be written (by taking into account that covariant derivatives commute),

$$[\hat{A}_t\,,\ \hat{A}_x]\,(g^{-1}Q) =\ 0.$$

After a somewhat lengthy calculation, and by using the operator identity

$$\begin{aligned}\hat{A}_t D_t + \hat{A}_x D_x &= D_t\hat{A}_t + D_x\hat{A}_x - \ [F\,[g]\ ,\ \ ]\\ &= D_t\hat{A}_t + D_x\hat{A}_x\ \ mod\ F\,[g]\end{aligned}$$

we find that the above integrability condition yields the PDE

$$(D_t\hat{A}_t +\ D_x\hat{A}_x)\,(g^{-1}Q')\ =\ 0\ \ mod\ F\,[g]$$

which is just the symmetry condition (19) for Q'. We conclude that Eq.(20) is indeed an auto-BT for the aforementioned symmetry condition. This BT is equivalent to a recursion operator for symmetries of the field equation (18). It can be rewritten in the form (13), as follows:

$$\begin{aligned}\hat{A}_t\left(g^{-1}Q^{(n)}\right) &=\ D_x\ \left(g^{-1}Q^{(n+1)}\right)\\ \hat{A}_x\left(g^{-1}Q^{(n)}\right) &=\ \ -\ D_t\ \left(g^{-1}Q^{(n+1)}\right)\end{aligned}\tag{21}$$

(n= 0, ±1, ±2, ...). The conservation laws of the form (14) (which form a doubly infinite set) are written, in this case,

$$(D_t\hat{A}_t +\ D_x\hat{A}_x)\ \left(g^{-1}Q^{(n)}\right)\ =\ 0\ \ mod\ F\,[g]\tag{22}$$

(where all conserved "charges" $Q^{(n)}$ are symmetry characteristics), while the Lax pair (17) reads,

$$D_x\,(g^{-1}\Psi)\ \ =\ \lambda\,\hat{A}_t\,(g^{-1}\Psi)\ \ ,\qquad D_t\,(g^{-1}\Psi)\ \ =\ -\lambda\,\hat{A}_x\,(g^{-1}\Psi)\tag{23}$$

The proof of the Lax-pair property of the linear system (23) is sketched as follows: By the integrability condition $(g^{-1}\Psi)_{xt}\ =\ (g^{-1}\Psi)_{tx}$, we get:

$$S(\Psi\,;\,g)\ \equiv\ (D_t\hat{A}_t +\ D_x\hat{A}_x)\,(g^{-1}\Psi)\ =\ 0\tag{24}$$

On the other hand, the integrability condition $\lambda\,[\hat{A}_t\,,\ \hat{A}_x]\,(g^{-1}\Psi) =\ 0$, yields:

$$S\,(\Psi\,;\,g)\ -\ \ \left[F\,[g]\ ,\ g^{-1}\Psi\right]\ \ =\ 0$$

which, in view of Eq.(24), becomes $[F\,[g]\ ,\ g^{-1}\Psi]\ \ =\ 0$. This is valid independently of Ψ if $F\ [g]$=0, i.e., if g is a solution of the field equation (18). We conclude that the linear system (23) is indeed a Lax pair for the nonlinear PDE (18), the solution Ψ of which pair is a symmetry characteristic [as follows from Eq.(24)]. We note that this Lax pair is different from that found several years ago by Zakharov and Mikhailov [9].

We conclude this section by giving an example of using the BT (21) to find conserved charges $Q^{(n)}$. Let us consider the symmetry characteristic $Q^{(0)} = gM$, where M is an arbitrary constant matrix. The BT (21) with n=0, integrated for $Q^{(1)}$, yields

$$Q^{(1)} = \ g\,[X,\, M],$$

where X is the potential of Eq.(18), defined by the system of equations

$$g^{-1}g_t = X_x \ \ , \quad g^{-1}g_x = -X_t \tag{25}$$

We note that $Q^{(1)}$ is the characteristic of a *potential symmetry* [3,6]. Higher-order charges $Q^{(n)}$ with n>1 (which also are higher-order potential symmetries) are similarly found by recursive integration of the BT (21) with n= 1, 2, etc.
To find negatively-indexed charges and corresponding symmetries, we begin with the BT (21) with n=−1, which we integrate for $Q^{(-1)}$. The result is a rather uninteresting local symmetry: $Q^{(-1)} = \Lambda g$, where Λ is any constant matrix. Iterating for n=-2, however, we find a new characteristic $Q^{(-2)}$, given by the system of equations

$$Q_t - \ Q\,g^{-1}g_t = g\;(g^{-1}\Lambda g)_x \ \ , \quad Q_x - \ Q\,g^{-1}g_x = -\,g\;(g^{-1}\Lambda g)_t$$

(where we have put $Q^{(-2)} = Q$, for brevity). Higher-order, negatively-indexed charges are obtained by further iteration.
Unfortunately, in contrast to the "internal" symmetries considered above, the local coordinate symmetries [such as $Q^{(0)} = g_t$, $Q^{(0)} = g_x$, etc.] do not yield any new results by applying the BT (21). These latter symmetries, however, play an equally important role as internal ones in problems in more than two dimensions, as the example discussed in the next section will show.

5. Self-Dual Yang-Mills Equation

The self-dual Yang-Mills (SDYM) equation is written in the form

$$F\,[J] \ \equiv \ (J^{-1}J_y)_{\bar{y}} + \ (J^{-1}J_z)_{\bar{z}} \ = \ 0 \tag{26}$$

where J is assumed $SL(N,C)$-valued (i.e., det J=1). The four independent variables (appearing as subscripts) are constructed from the coordinates of an underlying complexified Euclidean space in such a way that $\bar{y}$ and $\bar{z}$ become the complex conjugates of y and z, respectively, when the above space is real. As usual, subscripts denote total derivatives with respect to these variables.
Let $\delta J = \alpha\, Q\,[J]$ be an infinitesimal symmetry of Eq.(26), where the characteristic Q is subject to the condition that $tr\,(J^{-1}Q) = 0$, required for producing new $SL(N,C)$ solutions from old ones. The symmetry condition is, in analogy with Eq.(19),

$$S(Q\,;\,J) \ \equiv \ (D_{\bar{y}}\hat{A}_y + D_{\bar{z}}\hat{A}_z)\,(J^{-1}Q) \ = \ 0 \ \ mod\,F\,[J] \tag{27}$$

where we have introduced the covariant derivatives

$$\hat{A}_y = D_y + [J^{-1}J_y\,,\quad]\ ,\qquad \hat{A}_z = D_z + [J^{-1}J_z\,,\quad]$$

(note again that these operators commute). An auto-BT for the linear PDE (27) [analogous to that of Eq.(20)], which is consistent with the physical requirement $tr\,(J^{-1}Q) = 0$, is the following:

$$\hat{A}_y(J^{-1}Q) = (J^{-1}Q')_{\bar{z}}\ ,\qquad \hat{A}_z(J^{-1}Q) = -(J^{-1}Q')_{\bar{y}} \tag{28}$$

Integrability for Q' requires that Q satisfy Eq.(27). Integrability for Q , expressed by the condition $[\hat{A}_y\,,\ \hat{A}_z]\,(J^{-1}Q) = 0$, and upon using the operator identity

$$\begin{aligned}\hat{A}_y D_{\bar{y}} + \hat{A}_z D_{\bar{z}} &= D_{\bar{y}}\hat{A}_y + D_{\bar{z}}\hat{A}_z - [F\,[J]\ ,\ \ ] \\ &= D_{\bar{y}}\hat{A}_y + D_{\bar{z}}\hat{A}_z\ \ mod\ F\,[J]\end{aligned}$$

leads us again to Eq.(27), this time for Q'. The BT (28) may be regarded as an invertible recursion operator for the SDYM equation. It can be re-expressed as

$$\begin{aligned}\hat{A}_y\left(J^{-1}Q^{(n)}\right) &= D_{\bar{z}}\left(J^{-1}Q^{(n+1)}\right) \\ \hat{A}_z\left(J^{-1}Q^{(n)}\right) &= -D_{\bar{y}}\left(J^{-1}Q^{(n+1)}\right)\end{aligned} \tag{29}$$

(n= 0, ±1, ±2, ...). From this we get a double infinity of conservation laws of the form

$$(D_{\bar{y}}\hat{A}_y + D_{\bar{z}}\hat{A}_z)\left(J^{-1}Q^{(n)}\right) = 0\ \ mod\ F\,[J] \tag{30}$$

Finally, the Lax pair for SDYM [analogous to those of Eqs.(17) and (23)] is

$$D_{\bar{z}}\,(J^{-1}\Psi) = \lambda\,\hat{A}_y\,(J^{-1}\Psi)\ ,\qquad D_{\bar{y}}\,(J^{-1}\Psi) = -\lambda\,\hat{A}_z\,(J^{-1}\Psi) \tag{31}$$

The proof of the Lax-pair property is sketched as follows: By the integrability condition $(J^{-1}\Psi)_{\bar{z}\bar{y}} - (J^{-1}\Psi)_{\bar{y}\bar{z}} = 0$, we get:

$$S(\Psi\,;\,J) \equiv (D_{\bar{y}}\hat{A}_y + D_{\bar{z}}\hat{A}_z)\,(J^{-1}\Psi) = 0 \tag{32}$$

On the other hand, the integrability condition $\lambda\,[\hat{A}_z\,,\ \hat{A}_y]\,(J^{-1}\Psi) = 0$, yields:

$$S\,(\Psi\,;\,J) - \left[F\,[J]\ ,\ J^{-1}\Psi\right] = 0$$

which, in view of Eq.(32), becomes $[F\,[J]\ ,\ J^{-1}\Psi] = 0$. This is valid independently of Ψ if $F\ [J\ ]$=0, i.e., if J is an SDYM solution. We conclude that the linear system (31) is indeed a Lax pair for the SDYM equation (26), the solution Ψ of which pair is a symmetry characteristic [as follows from Eq.(32)]. This Lax pair can be shown to be equivalent to that reported previously by this author [5], although the systematic method for explicitly constructing this system is presented here for the first time.

We now give examples of using the BT (29) to find conserved charges $Q^{(n)}$. Let us consider the symmetry characteristic $Q^{(0)} = JM$, where M is a constant traceless matrix. The BT (29) with n=0, integrated for $Q^{(1)}$, yields

$$Q^{(1)} = J\,[X,\, M],$$

where X is the potential of Eq.(26), defined by the system of equations

$$J^{-1}J_y = X_{\bar{z}} \;\;, \qquad J^{-1}J_z = -X_{\bar{y}} \tag{33}$$

We note that $Q^{(1)}$ is the characteristic of a potential symmetry [3,6]. Higher-order charges $Q^{(n)}$ with n>1 (which also are higher-order potential symmetries) are similarly found by recursive integration of the BT (29) with n= 1, 2, etc.
To find negatively-indexed charges and corresponding symmetries, we begin with the BT (29) with n=−1, which we integrate for $Q^{(-1)}$. The result is a familiar local symmetry: $Q^{(-1)} = \Lambda\, J$, where Λ is any constant traceless matrix. Iterating for n=-2, we find a new characteristic $Q^{(-2)}$, given by the system of equations

$$Q_y - \; Q\,J^{-1}J_y = J\,(J^{-1}\Lambda\,J)_{\bar{z}} \;\;, \qquad Q_z - \; Q\,J^{-1}J_z = -\,J\,(J^{-1}\Lambda\,J)_{\bar{y}}$$

(where we have put $Q^{(-2)} = Q$, for brevity). Higher-order, negatively-indexed charges are obtained by further iteration.
In the preceding example, the initial symmetry characteristic $Q^{(0)}$ represented an "internal" symmetry (a symmetry in the fiber space). Local coordinate symmetries (symmetries in the base space), however, also lead to the discovery of infinite sets of potential symmetries and associated conservation laws for SDYM. As an example, consider the obvious symmetry of y-translation, represented by the characteristic $Q^{(0)} = J_y$. The BT (29) with n=0, integrated for $Q^{(1)}$, gives

$$Q^{(1)} = J\,X_y$$

[where X is the SDYM potential defined in Eq.(33)], which is another potential symmetry. Higher-order potential symmetries, whose characteristics $Q^{(n)}$ (n>0) appear as conserved charges in conservation laws of the form (30), can be found by repeated application of the recursion operator (29). The infinite sets of potential symmetries generated by coordinate transformations have been shown to possess a rich Lie-algebraic structure [10,11].
To conclude our example, let us find some negatively-indexed symmetries. The BT (29) with n=−1 and $Q^{(0)} = J_y$, integrated for $Q^{(-1)}$, gives: $Q^{(-1)} = J_{\bar{z}}$, which is the characteristic for the obvious $\bar{z}$-translational symmetry. The first nontrivial result is found for n=-2, yielding a characteristic $Q^{(-2)}$ which is defined by the set of equations

$$Q_y - \; Q\,J^{-1}J_y = J\,(J^{-1}J_{\bar{z}})_{\bar{z}} \;\;, \qquad Q_z - \; Q\,J^{-1}J_z = -\,J\,(J^{-1}J_{\bar{z}})_{\bar{y}}$$

(where we have put $Q^{(-2)} = Q$, for brevity).

Summary

Motivated by the results of [1] for the Ernst equation, we have proposed a general, non-Noetherian scheme for connecting symmetry and integrability properties of nonlinear PDEs in conservation-law form. We have shown that, by starting with the symmetry condition (which is itself a local conservation law for the associated nonlinear PDE), one may derive significant mathematical objects such as a recursion operator for symmetries, a Lax pair, and an infinite collection of (generally nonlocal) conservation laws. Such objects are usually sought by trial-and-error processes, thus any systematic technique for their discovery is useful.

The method was illustrated by using two physically significant examples, namely, the chiral field equation and the self-dual Yang-Mills (SDYM) equation. The latter PDE has been shown to constitute a prototype equation from which several other integrable PDEs are derived by reduction [12,13]. Thus, the results regarding SDYM may also prove useful for the study of other nonlinear problems.

Appendix: Total Derivatives and Fréchet Derivatives

To make this article as self-contained as possible, we define two key concepts that are being used, namely, the total derivative and the Fréchet derivative. The reader is referred to the extensive review article [14] by this author for more details. (It should be noted, however, that our present definition of the Fréchet derivative corresponds to the definition of the Lie derivative in that article. Since these two derivatives are *locally* indistinguishable, this discrepancy in terminology should not cause any concern mathematically.)

We consider the set of all PDEs of the form $F\,[u]=0$, where, for simplicity, the solutions u (which may be matrix-valued) are assumed to be functions of only two variables x and t : $u=u(x,t)$. In general, $F[u] \equiv F(x,\, t,\, u,\, u_x,\, u_t,\, u_{xx},\, u_{tt},\, u_{xt},\, \cdots)$. Geometrically, we say that the function F is defined in a *jet space* [2,15] with coordinates x, t, u, and as many partial derivatives of u as needed for the given problem. A solution of the PDE $F\,[u]=0$ is then a surface in this jet space.

Let $F\,[u]$ be a given function in the jet space. When differentiating such a function with respect to x or t, both implicit (through u) and explicit dependence of F on these variables must be taken into account. If u is a scalar quantity, we define the *total derivative operators* D_x and D_t as follows:

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots$$
$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots$$

(note that the operators $\partial/\partial x$ and $\partial/\partial t$ concern only the explicit dependence of F on x and t). If, however, u is matrix-valued, the above representation has only symbolic significance and cannot be used for actual calculations. We must therefore define the total derivatives D_x and D_t in more general terms.

We define a linear operator D_x, acting on functions $F\ [u]$ in the jet space and having the following properties:
1. On functions $f\ (x,\ t)$ in the base space,

$$D_x f(x,t) = \partial f/\partial x \ \equiv \ \partial_x f.$$

2. On functions $F\ [u]= u$ or $u_x,\ u_t$, etc., in the "fiber" space,
$D_x u = u_x\ ,\ \ D_x u_x = u_{xx}\ ,\ \ D_x u_t = u_{tx} = u_{xt}$, etc.
3. The operator D_x is a derivation on the algebra of all functions $F\ [u]$ in the jet space (i.e., the Leibniz rule is satisfied):

$$D_x\left(F\,[u]\,G\,[u]\right) = \left(D_x F\,[u]\right)\,G\,[u]\ + F\,[u]\,D_x G\,[u].$$

We similarly define the operator D_t. Extension to higher-order total derivatives is obvious (although these latter derivatives are no longer derivations, i.e., they do not satisfy the Leibniz rule). The following notation has been used in this article:

$$D_x F\,[u] \equiv F_x\,[u]\ ,\quad D_t F\,[u] \equiv F_t\,[u].$$

Finally, it can be shown that, for any matrix-valued functions A and B in the jet space, we have

$$(A^{-1})_x = -\,A^{-1}A_x A^{-1}\ \ ,\quad (A^{-1})_t = -\,A^{-1}A_t A^{-1}$$

and

$$D_x\,[A\,,\,B] = \ [A_x\,,\,B] + [A\,,\,B_x]\ \ ,\quad D_t\,[A\,,\,B] = \ [A_t\,,\,B] + [A\,,\,B_t]$$

where square brackets denote commutators.
Let now $\delta u \simeq \alpha\,Q\,[u]$ be an infinitesimal symmetry transformation (with characteristic $Q\ [u]$) for the PDE $F\ [u]$=0. We define the *Fréchet derivative* with respect to the characteristic Q as a linear operator Δ acting on functions $F\ [u]$ in the jet space and having the following properties:
1. On functions $f\ (x,\ t)$ in the base space,

$$\Delta\, f(x,t) = 0$$

(this is a consequence of our liberty to choose all our symmetries to be in "vertical" form [2,3]).
2. On $F\ [u]= u$,

$$\Delta\,u = Q\,[u].$$

3. The operator Δ commutes with total derivative operators of any order.
4. The Leibniz rule is satisfied:

$$\Delta\left(F\,[u]\,G\,[u]\right) = \left(\Delta F\,[u]\right)\,G\,[u]\ + F\,[u]\,\Delta G\,[u].$$

The following properties can be proven:

$$\Delta\, u_x = (\Delta\, u)_x = Q_x\,[u] \quad , \qquad \Delta\, u_t = (\Delta\, u)_t = Q_t\,[u]$$

$$\Delta\,(A^{-1}) = -\,A^{-1}(\Delta\, A)\; A^{-1} ; \Delta\,[A\,,\,B] = \;[\Delta A\,,\;B] + \;[A\,,\;\Delta B]$$

where A and B are any matrix-valued functions in the jet space.
If the solution u of the PDE is a scalar function (thus so is the characteristic Q), the Fréchet derivative with respect to Q admits a differential-operator representation of the form

$$\Delta \;=\; Q\frac{\partial}{\partial u} + \;Q_x\frac{\partial}{\partial u_x} + \;Q_t\frac{\partial}{\partial u_t} + \;Q_{xx}\frac{\partial}{\partial u_{xx}} + \;Q_{tt}\frac{\partial}{\partial u_{tt}} + \;Q_{xt}\frac{\partial}{\partial u_{xt}} + \cdots$$

Such representations, however, are not valid for PDEs in matrix form. In these cases we must resort to the general definition of the Fréchet derivative given above.
Finally, by using the Fréchet derivative, the symmetry condition for a PDE $F\;[u]$=0 can be expressed as follows [2,3]:

$$\Delta F\,[u] \;\;=\;\; 0 \;\;\; mod\, F\,[u].$$

This condition yields a linear PDE for the symmetry characteristic Q , of the form

$$S\,(Q\;;\;u) \;\;=\;\; 0 \;\;\; mod\, F\,[u].$$

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Article

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BÄCKLUND-TRANSFORMATION-RELATED RECURSION OPERATORS: APPLICATION TO THE SELF-DUAL YANG–MILLS EQUATION

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By using the self-dual Yang–Mills (SDYM) equation as an example, we study a method for relating symmetries and recursion operators of two partial differential equations connected to each other by a non-auto-Bäcklund transformation. We prove the Lie-algebra isomorphism between the symmetries of the SDYM equation and those of the potential SDYM (PSDYM) equation, and we describe the construction of the recursion operators for these two systems. Using certain known aspects of the PSDYM symmetry algebra, we draw conclusions regarding the Lie algebraic structure of the "potential symmetries" of the SDYM equation.

Keywords: Bäcklund transformations; recursion operators; self-dual Yang–Mills equation.

Mathematics Subject Classification: 35Q58, 35Q40

1. Introduction

Recursion operators are powerful tools for the study of symmetries of partial differential equations (PDEs). Roughly speaking, a recursion operator is a linear operator which produces a new symmetry characteristic of a PDE whenever it acts on an "old" characteristic (see Appendix). The concept was first introduced by Olver [1, 2] and subsequently used by many authors (see, e.g., [2, 3] and the references therein). An alternative view, based on the concept of a Bäcklund transformation (BT), was developed in a series of papers by the present authors [4–6] who studied the symmetry problem for the self-dual Yang–Mills equation (SDYM). The general idea is that a recursion operator can be viewed as an auto-BT for the "linearization equation" (or symmetry condition) of a (generally nonlinear) PDE.

This idea was later further developed and put into formal mathematical perspective by Marvan [7].

It has been known for some time (see, e.g., [3, Sec. 7.4] and the references therein) that, when two nonlinear PDEs are connected by a non-auto-BT, symmetries of either PDE may yield symmetries of the other. This can be achieved by using the original BT to construct another non-auto-BT which relates solutions of the linearization equations of the two PDEs. In the particular case of the SDYM equation, the original BT associates this PDE with the "potential SDYM equation" (PSDYM). The symmetries of the latter PDE can then be used to construct the "potential symmetries" of SDYM [5, 8]. We now attempt to go one step further: Can we find a BT which relates *recursion operators* of two PDEs? Given that, as said above, a recursion operator is itself an auto-BT, what we are after is a BT connecting two auto-BTs, each of which produces solutions of a respective linear PDE (symmetry condition). Thus, we are looking for "a transformation of transformations" rather than a transformation of functions.

Our "laboratory" model will again be SDYM, for good reasons. First, it possesses a rich symmetry structure; second, this PDE has been shown to constitute a sort of prototype equation from which several other integrable PDEs are derived by reduction (see, e.g., [9, 10]). By employing a non-auto-BT that connects SDYM with PSDYM, we will show how symmetries and recursion operators of either system can be associated with symmetries and recursion operators, respectively, of the other system. Moreover, we will prove that the symmetry Lie algebras of these two PDEs are isomorphic to each other. This conclusion is more than of academic importance, since it allows us to investigate the symmetry structure of the SDYM problem by studying the relatively easier PSDYM problem. As an example, we will recover the known infinite-dimensional symmetry algebras of SDYM [11–13] from the symmetry structure of PSDYM [8] and show how these algebras are related to potential symmetries.

2. The Symmetry Problem for the SDYM-PSDYM System

We write the SDYM equation in the form

$$F[J] \equiv D_{\bar{y}}(J^{-1}J_y) + D_{\bar{z}}(J^{-1}J_z) = 0. \tag{1}$$

We denote by $x^\mu \equiv y, z, \bar{y}, \bar{z}$ ($\mu = 1, \ldots, 4$) the independent variables, and by D_y, D_z, etc., the total derivatives with respect to these variables. We will also use the notation $D_y F \equiv F_y$, etc., for any function F. We assume that J is $SL(N, C)$-valued (i.e., $\det J = 1$).

We consider the non-auto-BT

$$J^{-1}J_y = X_{\bar{z}}, \quad J^{-1}J_z = -X_{\bar{y}}. \tag{2}$$

The integrability condition $(X_{\bar{y}})_{\bar{z}} = (X_{\bar{z}})_{\bar{y}}$ yields the SDYM equation (1). The integrability condition $(J_y)_z = (J_z)_y$, which is equivalent to

$$D_y(J^{-1}J_z) - D_z(J^{-1}J_y) + [J^{-1}J_y, J^{-1}J_z] = 0,$$

yields a nonlinear PDE for the "potential" X of (1), called the "potential SDYM equation" or PSDYM:

$$G[X] \equiv X_{y\bar{y}} + X_{z\bar{z}} - [X_{\bar{y}}, X_{\bar{z}}] = 0. \tag{3}$$

Noting that, according to (2), $(trX)_{\bar{z}} = [tr(\ln J)]_y = [\ln(\det J)]_y$, etc., we see that the condition $\det J = 1$ can be satisfied by requiring that $trX = 0$ [this requirement is compatible with the PSDYM equation (3)]. Hence, $SL(N,C)$ SDYM solutions correspond to $sl(N,C)$ PSDYM solutions.

Let $\delta J = \alpha Q$ and $\delta X = \alpha \Phi$ be an infinitesimal symmetry of system (2) (α is an infinitesimal parameter). This means that $(J + \delta J,\ X + \delta X)$ is a solution to the system when (J, X) is a solution. This suggests that the integrability conditions $F[J+\delta J] = 0$ and $G[X + \delta X] = 0$ are satisfied when the integrability conditions $F[J] = 0$ and $G[X] = 0$ are satisfied; that is, $J+\delta J$ and $X+\delta X$ are solutions of (1) and (3), respectively. The functions Q and Φ are *symmetry characteristics* for the above PDEs. Geometrically, the symmetries of system (2) are realized as transformations in the jet-like space of the variables $\{x^\mu, J, X\}$ and the various derivatives of J and X with respect to the x^μ. These transformations are generated by vector fields which, without loss of generality, may be considered "vertical", i.e., with vanishing projections on the base space of the x^μ [2]. We formally represent these vectors by differential operators of the form

$$V = Q\frac{\partial}{\partial J} + \Phi\frac{\partial}{\partial X}(+\ prolongation\ terms). \tag{4}$$

Consider a function $M(J, X)$. Denote by $\Delta M(J, X)$ the Fréchet derivative [2] of M with respect to V. The infinitesimal variation of M in the "direction" of V is then $\delta M = \alpha \Delta M$. The linear operator Δ is a derivation on the algebra of all $gl(N,C)$-valued functions. The Leibniz rule is written

$$\Delta(MN) = (\Delta M)N + M\Delta N. \tag{5}$$

In particular, for the Lie algebra of $sl(N,C)$-valued functions, the Leibniz rule may also be written as

$$\Delta[M, N] = [\Delta M, N] + [M, \Delta N]. \tag{6}$$

By definition, the Fréchet derivatives of the fundamental variables J and X are given by

$$\Delta J = Q, \quad \Delta X = \Phi. \tag{7}$$

We also note that the Fréchet derivative with respect to a *vertical* vector field commutes with all total derivative operators [2]. Finally, for an invertible matrix M,

$$\Delta(M^{-1}) = -M^{-1}(\Delta M)M^{-1}. \tag{8}$$

(For a discussion of the general symmetry problem for matrix-valued PDEs, see [14].)

We introduce the covariant derivative operators (with square brackets denoting commutators):

$$\begin{aligned} \hat{A}_y &\equiv D_y + [J^{-1}J_y, \,] = D_y + [X_{\bar{z}}, \,] \\ \hat{A}_z &\equiv D_z + [J^{-1}J_z, \,] = D_z - [X_{\bar{y}}, \,] \end{aligned} \tag{9}$$

where the BT (2) has been taken into account. By using (3) and the Jacobi identity, the zero-curvature condition $[\hat{A}_y, \hat{A}_z] = 0$ is shown to be satisfied, as expected in view of the fact that the "connections" $J^{-1}J_y$ and $J^{-1}J_z$ are pure gauges. Moreover, the linear operators of (9) are derivations on the Lie algebra of $sl(N, C)$-valued functions, satisfying a Leibniz rule of the form (6):

$$\begin{aligned} \hat{A}_y[M,N] &= [\hat{A}_y M, N] + [M, \hat{A}_y N] \\ \hat{A}_z[M,N] &= [\hat{A}_z M, N] + [M, \hat{A}_z N]. \end{aligned} \tag{10}$$

If Eqs. (1)–(3) are satisfied, then so must be their Fréchet derivatives with respect to the symmetry vector field V of (4). We now derive the symmetry condition for each of the above three systems. For SDYM (1), the symmetry condition is $\Delta F[J] = 0$, or

$$D_{\bar{y}}\Delta(J^{-1}J_y) + D_{\bar{z}}\Delta(J^{-1}J_z) = 0 \tag{11}$$

(since the Fréchet derivative Δ commutes with total derivatives). By using (5), (7), (8) and (9), it can be shown that

$$\Delta(J^{-1}J_y) = \hat{A}_y(J^{-1}Q), \quad \Delta(J^{-1}J_z) = \hat{A}_z(J^{-1}Q). \tag{12}$$

The SDYM symmetry condition (11) then becomes

$$(D_{\bar{y}}\hat{A}_y + D_{\bar{z}}\hat{A}_z)(J^{-1}Q) = 0. \tag{13}$$

The symmetry condition for PSDYM (3) is $\Delta G[X] = 0$, or, by using (6), (7) and (9),

$$\hat{A}_y\Phi_{\bar{y}} + \hat{A}_z\Phi_{\bar{z}} \equiv (\hat{A}_y D_{\bar{y}} + \hat{A}_z D_{\bar{z}})\Phi = 0. \tag{14}$$

We note the operator identity

$$\hat{A}_y D_{\bar{y}} + \hat{A}_z D_{\bar{z}} = D_{\bar{y}}\hat{A}_y + D_{\bar{z}}\hat{A}_z \tag{15}$$

which is easily verified by letting the right-hand side act on an arbitrary function M. Then, (14) is written in the alternate form,

$$(D_{\bar{y}}\hat{A}_y + D_{\bar{z}}\hat{A}_z)\Phi = 0. \tag{16}$$

Comparing (13) and (16), we observe that the symmetry characteristic Φ of PSDYM, and the function $J^{-1}Q$, where Q is an SDYM symmetry characteristic, satisfy the same symmetry condition. We thus conclude the following (see also [5]):

- If Q is an SDYM characteristic, then $\Phi = J^{-1}Q$ is a PSDYM characteristic. Conversely,
- If Φ is a PSDYM characteristic, then $Q = J\Phi$ is an SDYM characteristic.

Finally, the Fréchet derivative with respect to V also leaves the system of PDEs (2) invariant: $\Delta(J^{-1}J_y) = (\Delta X)_{\bar{z}}, \Delta(J^{-1}J_z) = -(\Delta X)_{\bar{y}}$. With the aid of (12) and (7) we are thus led to a pair of PDEs,

$$\hat{A}_y(J^{-1}Q) = \Phi_{\bar{z}}, \quad \hat{A}_z(J^{-1}Q) = -\Phi_{\bar{y}}. \tag{17}$$

Equation (17) is a BT connecting the symmetry characteristic Φ of PSDYM with the symmetry characteristic Q of SDYM. Indeed, the integrability condition $(\Phi_{\bar{z}})_{\bar{y}} = (\Phi_{\bar{y}})_{\bar{z}}$ yields the symmetry condition (13) for SDYM. So, when Q is an SDYM symmetry characteristic, the BT (17) is integrable for Φ. Conversely, the integrability condition $[\hat{A}_z, \hat{A}_y](J^{-1}Q) = 0$, valid in view of the zero-curvature condition, yields the PSDYM symmetry condition (14) for Φ and guarantees integrability for Q.

We note that, for a given Q, the solution of the BT (17) for Φ is not unique, and vice versa. To achieve uniqueness we thus need to make some additional assumptions: (a) If Φ is a solution for a given Q, then so is $\Phi + M(y,z)$, where M is an arbitrary matrix function. We make the agreement that any arbitrary additive term of the form $M(y,z)$ will be ignored when it appears in the solution for Φ. (b) If Q is a solution for a given Φ, then so is $Q + \varepsilon(\bar{y},\bar{z})J$, where $\varepsilon(\bar{y},\bar{z})$ is an arbitrary matrix function. We agree that any arbitrary additive term of the form $\varepsilon(\bar{y},\bar{z})J$ will be ignored when it appears in the solution for Q.

With the above conventions, the BT (17) establishes a 1-1 correspondence between the symmetries of SDYM and those of PSDYM. In particular, the SDYM characteristic $Q = 0$ corresponds to the PSDYM characteristic $\Phi = 0$. It will be shown below that this correspondence between the two symmetry sets is a Lie algebra isomorphism.

3. Recursion Operators and Lie-Algebra Isomorphism

Since the two PDEs in (17) are consistent with each other and solvable for Φ when Q is an SDYM symmetry characteristic, we may use, say, the first equation to formally express Φ in terms of Q:

$$\Phi = D_{\bar{z}}^{-1}\hat{A}_y(J^{-1}Q) \equiv \hat{R}(J^{-1}Q) \tag{18}$$

where we have introduced the linear operator

$$\hat{R} = D_{\bar{z}}^{-1}\hat{A}_y. \tag{19}$$

Proposition 1. *The operator* (19) *is a recursion operator for PSDYM.*

Proof. Let Φ be a symmetry characteristic for PSDYM. Then, Φ satisfies the symmetry conditions (14) or (16). We will show that $\Phi' \equiv \hat{R}\Phi$ also is a symmetry characteristic. Indeed,

$$\begin{aligned}(\hat{A}_y D_{\bar{y}} + \hat{A}_z D_{\bar{z}})\Phi' &\equiv (\hat{A}_y D_{\bar{y}} + \hat{A}_z D_{\bar{z}})\hat{R}\Phi \\ &= \hat{A}_y D_{\bar{z}}^{-1} D_{\bar{y}}\hat{A}_y\Phi + \hat{A}_z\hat{A}_y\Phi \\ &= \hat{A}_y D_{\bar{z}}^{-1}(D_{\bar{y}}\hat{A}_y + D_{\bar{z}}\hat{A}_z)\Phi + [\hat{A}_z,\hat{A}_y]\Phi = 0,\end{aligned}$$

in view of (16) and the zero-curvature condition $[\hat{A}_y,\hat{A}_z] = 0$. □

For $sl(N,C)$ PSDYM solutions, the symmetry characteristic Φ must be traceless. Then, so is the characteristic $\Phi' = \hat{R}\Phi$. That is, the recursion operator (19) preserves the $sl(N,C)$ character of PSDYM.

Is there a systematic process by which one could *derive* the recursion operator (19)? To this end, we seek an auto-BT relating solutions of the PSDYM symmetry condition (14). As shown in [5], such a BT is

$$\hat{A}_y\Phi = \Phi'_{\bar{z}}, \quad \hat{A}_z\Phi = -\Phi'_{\bar{y}}. \tag{19a}$$

The first of these equations can then be re-expressed as $\Phi' = \hat{R}\Phi$, with $\hat{R}$ given by (19).

Consider now a symmetry characteristic Q of SDYM, i.e., a solution of the symmetry condition (13). Also, consider the transformation

$$Q' = J\hat{R}(J^{-1}Q) \equiv \hat{T}Q \tag{20}$$

where we have introduced the linear operator

$$\hat{T} = J\hat{R}J^{-1}. \tag{21}$$

Proposition 2. *The operator* (21) *is a recursion operator for SDYM.*

Proof. By assumption, Q is an SDYM symmetry characteristic. Then, as shown above, $\Phi = J^{-1}Q$ is a PSDYM characteristic. Since $\hat{R}$ is a PSDYM recursion operator, $\Phi' \equiv \hat{R}\Phi = \hat{R}(J^{-1}Q)$ also is a PSDYM characteristic. Then, finally, $Q' = J\Phi'$, given by (20), is an SDYM characteristic. □

For $SL(N, C)$ SDYM solutions, the symmetry characteristic Q must satisfy the condition $tr(J^{-1}Q) = 0$. As can be seen, this condition is preserved by the recursion operator (21). [Note, in this connection, that the BT (17) or (18) properly associates SL(N, C) SDYM characteristics Q with *sl*(N, C) PSDYM characteristics Φ.]

The recursion operator (21) also can be derived from an auto-BT for the SDYM symmetry condition (13). This BT was constructed in [6] by using a properly chosen Lax pair for SDYM (we refer the reader to this paper for details). We may thus conclude that recursion operators such as (19) or (21) in effect represent auto-BTs for symmetry conditions of respective nonlinear PDEs (see also [7]).

Lemma. *The Fréchet derivative* Δ *with respect to the vector* V *of* (4), *and the recursion operator* $\hat{R}$ *of* (19), *satisfy the commutation relation*

$$[\Delta, \hat{R}] = D_{\bar{z}}^{-1}[\Phi_{\bar{z}},] \tag{22}$$

where $\Phi = \Delta X$, *according to* (7).

Proof. Introducing an auxiliary function F, and using the derivation property (6) of Δ and the commutativity of Δ with all total derivatives (as well as all powers of such derivatives), we have:

$$\begin{aligned}\Delta\hat{R}F &= \Delta D_{\bar{z}}^{-1}\hat{A}_y F = D_{\bar{z}}^{-1}\Delta(D_y F + [X_{\bar{z}}, F]) \\ &= D_{\bar{z}}^{-1}(D_y \Delta F + [(\Delta X)_{\bar{z}}, F] + [X_{\bar{z}}, \Delta F]) \\ &= D_{\bar{z}}^{-1}(\hat{A}_y \Delta F + [\Phi_{\bar{z}}, F]) = \hat{R}\Delta F + D_{\bar{z}}^{-1}[\Phi_{\bar{z}}, F],\end{aligned}$$

from which there follows (22). □

Proposition 3. *The BT* (17), *or equivalently, its solution* (18), *establishes an isomorphism between the symmetry Lie algebras of SDYM and PSDYM.*

Proof. Let V be a vector field of the form (4), generating a symmetry of the BT (2). As explained previously, since this BT is invariant under V, the same will be true with regard to its integrability conditions. Hence, V also represents a symmetry of the SDYM-PSDYM system of Eqs. (1) and (3). The SDYM and PSDYM characteristics are $Q = \Delta J$ and

$\Phi = \Delta X$, respectively, where Δ denotes the Fréchet derivative with respect to V. Consider the linear map I defined by (18):

$$I : \Phi = I\{Q\} = \hat{R}J^{-1}Q \tag{23}$$

or

$$I : \Delta X = I\{\Delta J\} = \hat{R}J^{-1}\Delta J. \tag{24}$$

Consider also a pair of symmetries of system (2), indexed by i and j. These are generated by vector fields $V^{(r)}$, where $r = i, j$. The Fréchet derivatives with respect to the $V^{(r)}$ will be denoted $\Delta^{(r)}$. The SDYM and PSDYM symmetry characteristics are $Q^{(r)} = \Delta^{(r)}J$ and $\Phi^{(r)} = \Delta^{(r)}X$, respectively. According to (24),

$$\Delta^{(r)}X = I\{\Delta^{(r)}J\} = \hat{R}J^{-1}\Delta^{(r)}J = \hat{R}J^{-1}Q^{(r)}; \quad r = i, j. \tag{25}$$

By the Lie-algebraic property of symmetries of PDEs, the functions $[\Delta^{(i)}, \Delta^{(j)}]J$ and $[\Delta^{(i)}, \Delta^{(j)}]X$ also represent symmetry characteristics for SDYM and PSDYM, respectively, where we have put

$$\begin{aligned}
[\Delta^{(i)}, \Delta^{(j)}]J &\equiv \Delta^{(i)}\Delta^{(j)}J - \Delta^{(j)}\Delta^{(i)}J = \Delta^{(i)}Q^{(j)} - \Delta^{(j)}Q^{(i)},\\
[\Delta^{(i)}, \Delta^{(j)}]X &\equiv \Delta^{(i)}\Delta^{(j)}X - \Delta^{(j)}\Delta^{(i)}X = \Delta^{(i)}\Phi^{(j)} - \Delta^{(j)}\Phi^{(i)}.
\end{aligned}$$

We must now verify that

$$[\Delta^{(i)}, \Delta^{(j)}]X = I\{[\Delta^{(i)}, \Delta^{(j)}]J\} = \hat{R}J^{-1}[\Delta^{(i)}, \Delta^{(j)}]J. \tag{26}$$

Putting $r = j$ into (25), and applying the Fréchet derivative $\Delta^{(i)}$, we have:

$$\begin{aligned}
\Delta^{(i)}\Delta^{(j)}X = \Delta^{(i)}\hat{R}J^{-1}Q^{(j)} &= [\Delta^{(i)}, \hat{R}]J^{-1}Q^{(j)} + \hat{R}\Delta^{(i)}J^{-1}Q^{(j)}\\
&= D_{\bar{z}}^{-1}[\Phi_{\bar{z}}^{(i)}, J^{-1}Q^{(j)}] + \hat{R}\Delta^{(i)}J^{-1}Q^{(j)},
\end{aligned}$$

where we have used the commutation relation (22). By (23) and (19),

$$\Phi_{\bar{z}}^{(i)} = D_{\bar{z}}\hat{R}J^{-1}Q^{(i)} = \hat{A}_y J^{-1}Q^{(i)}.$$

Moreover, by properties (5) and (8) of the Fréchet derivative,

$$\begin{aligned}
\Delta^{(i)}J^{-1}Q^{(j)} &= -J^{-1}(\Delta^{(i)}J)J^{-1}Q^{(j)} + J^{-1}\Delta^{(i)}Q^{(j)}\\
&= -J^{-1}Q^{(i)}J^{-1}Q^{(j)} + J^{-1}\Delta^{(i)}Q^{(j)}.
\end{aligned}$$

So,

$$\Delta^{(i)}\Delta^{(j)}X = D_{\bar{z}}^{-1}[\hat{A}_y J^{-1}Q^{(i)}, J^{-1}Q^{(j)}] - \hat{R}J^{-1}Q^{(i)}J^{-1}Q^{(j)} + \hat{R}J^{-1}\Delta^{(i)}Q^{(j)}.$$

Subtracting from this the analogous expression for $\Delta^{(j)}\Delta^{(i)}X$, we have:

$$\begin{aligned}
[\Delta^{(i)}, \Delta^{(j)}]X &\equiv \Delta^{(i)}\Delta^{(j)}X - \Delta^{(j)}\Delta^{(i)}X\\
&= D_{\bar{z}}^{-1}([\hat{A}_y J^{-1}Q^{(i)}, J^{-1}Q^{(j)}] + [J^{-1}Q^{(i)}, \hat{A}_y J^{-1}Q^{(j)}])\\
&\quad - \hat{R}[J^{-1}Q^{(i)}, J^{-1}Q^{(j)}] + \hat{R}J^{-1}(\Delta^{(i)}Q^{(j)} - \Delta^{(j)}Q^{(i)})
\end{aligned}$$

$$
\begin{aligned}
&= D_{\bar{z}}^{-1} \hat{A}_y [J^{-1} Q^{(i)}, J^{-1} Q^{(j)}] - \hat{R} [J^{-1} Q^{(i)}, J^{-1} Q^{(j)}] \\
&\quad + \hat{R} J^{-1} (\Delta^{(i)} \Delta^{(j)} J - \Delta^{(j)} \Delta^{(i)} J) \\
&= \hat{R} J^{-1} [\Delta^{(i)}, \Delta^{(j)}] J
\end{aligned}
$$

where we have used the derivation property (10) of $\hat{A}_y$ and we have taken (19) into account. Thus, (26) has been proven. □

Now, suppose $\hat{P}$ is a recursion operator for SDYM, while $\hat{S}$ is a recursion operator for PSDYM. Thus, if Q and Φ are symmetry characteristics for SDYM and PSDYM, respectively, then $Q' = \hat{P} Q$ and $\Phi' = \hat{S} \Phi$ also are symmetry characteristics.

Definition. The linear operators $\hat{P}$ and $\hat{S}$ will be called *equivalent with respect to the isomorphism* I (or I*-equivalent*) if the following condition is satisfied:

$$\hat{S} \Phi = I\{\hat{P} Q\} \quad \text{when } \Phi = I\{Q\}. \tag{27}$$

By using (23), the above condition is written

$$\hat{S} \Phi = \hat{R} J^{-1} \hat{P} Q \quad \text{when } \Phi = \hat{R} J^{-1} Q \Rightarrow \hat{S} \hat{R} J^{-1} Q = \hat{R} J^{-1} \hat{P} Q.$$

Thus, in order that $\hat{P}$ and $\hat{S}$ be I-equivalent recursion operators, the following operator equation must be satisfied on the infinite-dimensional linear space of all SDYM symmetry characteristics:

$$\hat{S} \hat{R} J^{-1} = \hat{R} J^{-1} \hat{P}. \tag{28}$$

Having already found a PSDYM recursion operator $\hat{S} = \hat{R}$, we now want to evaluate the I-equivalent SDYM recursion operator $\hat{P}$. To this end, we put $\hat{S} = \hat{R}$ in (28) and write

$$\hat{R} (\hat{R} J^{-1} - J^{-1} \hat{P}) = 0.$$

As is easy to see, this is satisfied for $\hat{P} = \hat{T}$, in view of (21). We thus conclude that

- The recursion operators $\hat{R}$ and $\hat{T}$, defined by (19) and (21), are I-equivalent.

We note that (28) is a sort of BT relating recursion operators of different PDEs, rather than solutions or symmetries of these PDEs. Thus, if a recursion operator is known for either PDE, this BT will yield a corresponding operator for the other PDE. Note that we have encountered BTs at various levels: (*a*) The non-auto-BT (2), relating solutions of two different nonlinear PDEs (1) and (3); (*b*) the BT (17), or equivalently (18), relating symmetry characteristics of these PDEs; (*c*) the recursion operators (19) and (21), which can be re-expressed as auto-BTs for the symmetry conditions (14) and (13), respectively; and (*d*) the BT (28), relating recursion operators for the original, nonlinear PDEs. (We make the technical observation that the first three BTs are "strong", while the last one is "weak"; see Appendix.)

Example. Consider the PSDYM symmetry characteristic $\Phi = X_z$ (z-translation). To find the I-related SDYM characteristic Q, we use (23):

$$\begin{aligned}\hat{R}J^{-1}Q = \Phi \Rightarrow D_{\bar{z}}^{-1}\hat{A}_y(J^{-1}Q) = X_z \Rightarrow \hat{A}_y(J^{-1}Q) = X_{z\bar{z}} \\ \overset{(2)}{\Rightarrow} (J^{-1}Q)_y + [J^{-1}J_y, J^{-1}Q] = (J^{-1}J_y)_z,\end{aligned}$$

which is satisfied for $Q = J_z$. By applying the recursion operator $\hat{T}$ on Q,

$$\begin{aligned}Q' = \hat{T}Q = J\hat{R}J^{-1}Q = JD_{\bar{z}}^{-1}\hat{A}_y(J^{-1}J_z) = JD_{\bar{z}}^{-1}\{(J^{-1}J_z)_y + [J^{-1}J_y, J^{-1}J_z]\} \\ = JD_{\bar{z}}^{-1}(J^{-1}J_y)_z \overset{(2)}{=} JD_{\bar{z}}^{-1}X_{z\bar{z}} = JX_z.\end{aligned}$$

To find the I-related PSDYM characteristic Φ', we use (23) once more:

$$\Phi' = \hat{R}J^{-1}Q' = \hat{R}X_z = \hat{R}\Phi.$$

We notice that $\hat{R}\Phi = I\{\hat{T}Q\}$ when $\Phi = I\{Q\}$, as expected by the fact that $\hat{R}$ and $\hat{T}$ are I-equivalent recursion operators.

Now, let $Q^{(0)}$ be some SDYM symmetry characteristic. By repeated application of the recursion operator $\hat{T}$, we obtain an infinite sequence of such characteristics:

$$Q^{(1)} = \hat{T}Q^{(0)}, \quad Q^{(2)} = \hat{T}Q^{(1)} = \hat{T}^2Q^{(0)}, \ldots, Q^{(n)} = \hat{T}Q^{(n-1)} = \hat{T}^nQ^{(0)}, \ldots$$

(we note that any power of a recursion operator also is a recursion operator). Also, let

$$\Phi^{(0)} = I\{Q^{(0)}\} = \hat{R}J^{-1}Q^{(0)} \tag{29}$$

be the PSDYM characteristic which is I-related to $Q^{(0)}$. Repeated application of the PSDYM recursion operator $\hat{R}$ will now yield an infinite sequence of PSDYM characteristics. Taking into account that $\hat{R}$ and $\hat{T}$ are I-equivalent recursion operators, we can write this sequence as follows:

$$\begin{aligned}&\Phi^{(1)} = \hat{R}\Phi^{(0)} = I\{\hat{T}Q^{(0)}\}, \quad \Phi^{(2)} = \hat{R}^2\Phi^{(0)} = I\{\hat{T}^2Q^{(0)}\}, \ldots, \\ &\Phi^{(n)} = \hat{R}^n\Phi^{(0)} = I\{\hat{T}^nQ^{(0)}\}, \ldots.\end{aligned}$$

Assume now that the infinite set of SDYM symmetries represented by the characteristics $\{Q^{(n)}\}$ ($n = 0, 1, 2, \ldots$) has the structure of a Lie algebra. This set then constitutes a symmetry subalgebra of SDYM. Given that the set $\{\Phi^{(n)}\}$ is I-related to $\{Q^{(n)}\}$ and that I is a Lie-algebra isomorphism, we conclude that the infinite set of characteristics $\{\Phi^{(n)}\}$ corresponds to a symmetry subalgebra of PSDYM which is isomorphic to the associated subalgebra $\{Q^{(n)}\}$ of SDYM.

More generally, let $\{Q_k^{(0)}/k = 1, 2, \ldots, p\}$ be a finite set of SDYM symmetry characteristics, and let $\{\Phi_k^{(0)}/k = 1, 2, \ldots, p\}$ be the I-related set of PSDYM characteristics, where

$$\Phi_k^{(0)} = I\{Q_k^{(0)}\} = \hat{R}J^{-1}Q_k^{(0)}; \quad k = 1, 2, \ldots, p. \tag{30}$$

Assume that the infinite set of characteristics

$$\{Q_k^{(n)} = \hat{T}^n Q_k^{(0)} / n = 0, 1, 2, \ldots ; k = 1, 2, \ldots , p\} \tag{31}$$

corresponds to a Lie subalgebra of SDYM symmetries. Then, the I-related set of characteristics

$$\{\Phi_k^{(n)} = \hat{R}^n \Phi_k^{(0)} / n = 0, 1, 2, \ldots ; k = 1, 2, \ldots , p\} \tag{32}$$

corresponds to a PSDYM symmetry subalgebra which is isomorphic to that of (31).

Let us summarize our main conclusions:

- The infinite-dimensional symmetry Lie algebras of SDYM and PSDYM are isomorphic, the isomorphism I being defined by (23) or (24).
- The recursion operators $\hat{T}$ and $\hat{R}$, defined in (21) and (19), when applied to I-related symmetry characteristics [such as those in (29) or (30)], may generate isomorphic, infinite-dimensional symmetry subalgebras of SDYM and PSDYM, respectively.
- Since the structures of the symmetry Lie algebras of SDYM and PSDYM are similar, all results regarding the latter structure are also applicable to the SDYM case.

Comment. At this point the reader may wonder whether it is really necessary to go through the PSDYM symmetry problem in order to solve the corresponding SDYM problem. In principle, of course, the SDYM case can be treated on its own. In practice, however, it is easier to study the symmetry structure of PSDYM first and then take advantage of the isomorphism between this structure and that of SDYM. This statement is justified by the fact that the PSDYM recursion operator is considerably easier to handle compared to the corresponding SDYM operator. This property of the former operator is of great value in the interest of computational simplicity (in particular, for the purpose of deriving various commutation relations; cf. [8]).

4. Potential Symmetries and Current Algebras

We recall that every SDYM symmetry characteristic can be expressed as $Q = J\Phi$, where Φ is a PSDYM characteristic (we note that Φ is *not* I-related to Q). Let Φ be a characteristic which depends locally or nonlocally on X and/or various derivatives of X. By the BT (2), X must be an integral of J and its derivatives, and so this and its derivatives X_y and X_z are nonlocal in J. On the other hand, according to (2), the quantities $X_{\bar{y}}$ and $X_{\bar{z}}$ depend locally on J. Thus, in general, Φ can be local or nonlocal in J. In the case where Φ is *nonlocal* in J, we say that the characteristic $Q = J\Phi$ expresses a *potential symmetry* of SDYM [3, 5]. (See Appendix for a general definition of locality and nonlocality of symmetries.)

4.1. *Internal symmetries*

The PSDYM equation is generally invariant under a transformation of the form

$$\Delta^{(0)} X = \Phi^{(0)} = [X, M] \tag{33}$$

where M is any constant $sl(N, C)$ matrix. Since the characteristic $\Phi^{(0)}$ is nonlocal in J, the transformation

$$Q = J\Phi^{(0)} = J[X, M]$$

is a genuine potential symmetry of SDYM. Note that the SDYM characteristic which is I-related to $\Phi^{(0)}$ is not Q, but rather $Q^{(0)} = JM$, since we then have

$$\hat{R}J^{-1}Q^{(0)} = \hat{R}M = D_{\bar{z}}^{-1}[X_{\bar{z}}, M] = [X, M] = \Phi^{(0)}.$$

Let $\{\tau_k\}$ be a basis for $sl(N, C)$:

$$[\tau_i, \tau_j] = C_{ij}^k \tau_k.$$

Then M is expanded as $M = \alpha^k \tau_k$, and (33) is resolved into a set of independent basis transformations

$$\Delta_k^{(0)} X = \Phi_k^{(0)} = [X, \tau_k]$$

corresponding to the SDYM potential symmetries

$$Q_k = J\Phi_k^{(0)} = J[X, \tau_k].$$

These are not the same as the I-related characteristics

$$\Delta_k^{(0)} J = Q_k^{(0)} = J\tau_k.$$

Consider now the infinite set of transformations

$$\Delta_k^{(n)} X = \Phi_k^{(n)} = \hat{R}^n \Phi_k^{(0)} = \hat{R}^n [X, \tau_k] \quad (n = 0, 1, 2, \ldots) \tag{34}$$

As can be shown, they satisfy the commutation relations of a Kac–Moody algebra:

$$[\Delta_i^{(m)}, \Delta_j^{(n)}]X = C_{ij}^k \Delta_k^{(m+n)} X.$$

In view of the isomorphism I, this structure is also present in SDYM. Indeed, this is precisely the familiar hidden symmetry of SDYM [11, 12]. The SDYM transformations which are I-related to those in (34) are given by

$$\Delta_k^{(n)} J = Q_k^{(n)} = \hat{T}^n Q_k^{(0)} = \hat{T}^n J\tau_k \quad (n = 0, 1, 2, \ldots).$$

They constitute an infinite set of potential symmetries (note, for example, that $\Delta_k^{(1)} J = J[X, \tau_k] = J\Phi_k^{(0)}$) and they satisfy the commutation relations

$$[\Delta_i^{(m)}, \Delta_j^{(n)}]J = C_{ij}^k \Delta_k^{(m+n)} J.$$

4.2. *Symmetries in the base space*

A number of local PSDYM symmetries corresponding to coordinate transformations are nonlocal in J, hence lead to potential symmetries of SDYM. By using isovector methods

[4, 15], nine such PSDYM symmetries can be found. They can be expressed as follows:

$$\Delta_k^{(0)} X = \Phi_k^{(0)} = \hat{L}_k X \quad (k = 1, 2, \ldots, 9) \tag{35}$$

where the $\hat{L}_k$ are nine linear operators which are explicitly given by

$$\begin{aligned}
&\hat{L}_1 = D_y, \quad \hat{L}_2 = D_z, \quad \hat{L}_3 = zD_y - \bar{y}D_{\bar{z}}, \quad \hat{L}_4 = yD_z - \bar{z}D_{\bar{y}}, \\
&\hat{L}_5 = yD_y - zD_z - \bar{y}D_{\bar{y}} + \bar{z}D_{\bar{z}}, \quad \hat{L}_6 = 1 + yD_y + zD_z, \\
&\hat{L}_7 = 1 - \bar{y}D_{\bar{y}} - \bar{z}D_{\bar{z}}, \quad \hat{L}_8 = y\hat{L}_6 + \bar{z}(yD_{\bar{z}} - zD_{\bar{y}}), \\
&\hat{L}_9 = z\hat{L}_6 + \bar{y}(zD_{\bar{y}} - yD_{\bar{z}}).
\end{aligned}$$

The $\hat{L}_1, \hat{L}_2$ represent translations of y and z, respectively, while the $\hat{L}_3, \hat{L}_4$ represent rotational symmetries. The $\hat{L}_5, \hat{L}_6, \hat{L}_7$ express scale transformations, while $\hat{L}_8$ and $\hat{L}_9$ represent nonlinear coordinate transformations which presumably reflect the special conformal invariance of the SDYM equations in their original, covariant form.

The first five operators $\hat{L}_1, \ldots, \hat{L}_5$ form the basis of a Lie algebra, the commutation relations of which we write in the form

$$[\hat{L}_i, \hat{L}_j] = -f_{ij}^k \hat{L}_k \quad (k = 1, \ldots, 5).$$

Consider now the infinite set of transformations

$$\Delta_k^{(n)} X = \Phi_k^{(n)} = \hat{R}^n \Phi_k^{(0)} = \hat{R}^n \hat{L}_k X \quad (k = 1, \ldots, 5). \tag{36}$$

As can be shown [8], these form a Kac–Moody algebra:

$$[\Delta_i^{(m)}, \Delta_j^{(n)}] X = f_{ij}^k \Delta_k^{(m+n)} X.$$

Consider also the infinite sets of transformations

$$\Delta^{(n)} X = \hat{R}^n \hat{L}_6 X \quad \text{and} \quad \Delta^{(n)} X = \hat{R}^n \hat{L}_7 X. \tag{37}$$

As can be proven [8], each set forms a Virasoro algebra (apart from a sign):

$$[\Delta^{(m)}, \Delta^{(n)}] X = -(m - n) \Delta^{(m+n)} X.$$

Taking the isomorphism I into account, we conclude that the SDYM symmetry algebra possesses both Kac–Moody and Virasoro subalgebras ("current algebras" [16]), both of which are associated with infinite sets of potential symmetries. The former subalgebras are associated with both internal and coordinate transformations, while the latter ones are related to coordinate transformations only. These conclusions are in agreement with those of [13], although the mathematical approach there is different from ours.

5. Summary

By using the SDYM-PSDYM system as a model, we have studied a process for associating symmetries and recursion operators of two nonlinear PDEs related to each other by a non-auto-BT. The concept of a BT itself enters our analysis at various levels: (*a*) The non-auto BT (2) relates solutions of the nonlinear PDEs (1) and (3); (*b*) the non-auto-BT (17) or

(18) relates symmetry characteristics of these PDEs; (c) the auto-BTs for the symmetry conditions (14) and (13) lead to the recursion operators (19) and (21), respectively; and (d) the transformation (28) may be perceived as a BT associating recursion operators for the original, nonlinear PDEs. We have proven the isomorphism between the infinite-dimensional symmetry Lie algebras of SDYM and PSDYM, and we have used this property to draw several conclusions regarding the Lie-algebraic structure of the potential symmetries of SDYM.

For further reading on recursion operators, the reader is referred to [17–22]. A nice discussion of the SDYM symmetry structure and its connection to the existence of infinitely many conservation laws can be found in the paper by Adam *et al.* [23].

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Appendix: Some Basic Definitions

To make the paper as self-contained as possible, basic definitions of some key concepts that are being used are given below:

A.1. *Recursion operators*

Consider a PDE $F[u] = 0$, in the dependent variable u and the independent variables x^μ ($\mu = 1, 2, \ldots$). Let $\delta u = \alpha Q[u]$ be an infinitesimal symmetry transformation of the PDE, where $Q[u]$ is the *symmetry characteristic*. The symmetry is generated by the (formal) vector field

$$V = Q[u]\frac{\partial}{\partial u} + \textit{ prolongation } = Q\frac{\partial}{\partial u} + Q_\mu \frac{\partial}{\partial u_\mu} + Q_{\mu\nu}\frac{\partial}{\partial u_{\mu\nu}} + \cdots \tag{A.1}$$

(where the $Q_\mu \equiv D_\mu Q$, etc., denote total derivatives of Q). The symmetry condition is expressed by a PDE, linear in Q:

$$S(Q; u) \equiv \Delta F[u] = 0 \quad \mathrm{mod}\ F[u] \tag{A.2}$$

where Δ denotes the Fréchet derivative with respect to V. If u is a scalar quantity, then (A.2) takes the form

$$S(Q; u) = VF[u] = Q\frac{\partial F}{\partial u} + Q_\mu \frac{\partial F}{\partial u_\mu} + Q_{\mu\nu}\frac{\partial F}{\partial u_{\mu\nu}} + \cdots = 0 \quad \mathrm{mod}\ F[u]. \tag{A.3}$$

Since the PDE (A.2) is linear in Q, the sum of two solutions (for the same u) also is a solution. Thus, for any given u, the solutions $\{Q[u]\}$ of (A.2) form a linear space S_u. A *recursion operator* $\hat{R}$ is a linear operator which maps the space S_u into itself. Thus, if Q is a symmetry characteristic of $F[u] = 0$ [i.e., a solution of (A.2)], then so is $\hat{R}Q$:

$$S(\hat{R}Q; u) = 0 \quad \text{when } S(Q; u) = 0. \tag{A.4}$$

We note that $\hat{R}^2Q, \hat{R}^3Q, \ldots, \hat{R}^nQ$ also are symmetry characteristics. This means that

any power $\hat{R}^n$ *of a recursion operator also is a recursion operator.*

Thus, starting with any symmetry characteristic Q, we can obtain an infinite set of such characteristics by repeated application of the recursion operator.

A *symmetry operator* $\hat{L}$ is a linear operator, independent of u, which produces a symmetry characteristic $Q[u]$ when it acts on u. Thus, $\hat{L}u = Q[u]$. We note that $\hat{R}\hat{L}u$ is a symmetry characteristic, which means that

the product $\hat{R}\hat{L}$ of a recursion operator and a symmetry operator is a symmetry operator.

Thus, given that $\hat{R}^n$ is a recursion operator, we conclude that $\hat{R}^n\hat{L}u$ is a member of S_u. Examples of symmetry operators are the nine operators $\hat{L}_k$ that appear in (35), as well as the operator $\hat{L} = [\ , M]$ which is implicitly defined in (33).

A.2. *"Strong" and "Weak" Bäcklund transformations*

In the most general sense, a BT is a set of relations (typically differential, although in certain cases algebraic ones are also considered) which connect solutions of two different PDEs (non-auto-BT) or of the same PDE (auto-BT). The technical distinction between "strong" and "weak" BTs [24, 25] can be roughly described as follows: In a strong BT connecting, say, the variables u and v, integrability of the differential system for either variable *demands* that the other variable satisfy a certain PDE. A weak BT, on the other hand, is much like a symmetry transformation: u and v are not, *a priori*, required to satisfy any particular PDEs for integrability. *If*, however, u satisfies some specific PDE, *then* v satisfies some related PDE. (An example is the Cole–Hopf transformation, connecting solutions of Burgers' equation to solutions of the heat equation.)

The BT (2) is strong, since its integrability conditions *force* the functions J and X to satisfy the PDEs (1) and (3), respectively. Similar remarks apply to the BTs (17) and (19a). On the other hand, transformation (28) does not *a priori* impose any specific properties on the operators $\hat{P}$ and $\hat{S}$. *If*, however, $\hat{P}$ is an SDYM recursion operator, *then* $\hat{S}$ is the I-equivalent PSDYM recursion operator. Thus, equation (28) is a Bäcklund-like transformation of the weak type, although this particular transformation relates operators rather than functions.

A.3. *Local and nonlocal symmetries*

Let $F[u] = 0$ be a PDE in the dependent variable u and the independent variables x^μ ($\mu = 1, 2, \ldots$). A symmetry characteristic $Q[u]$ represents a *local* symmetry of the PDE if Q depends, at most, on x^μ, u, and derivatives of u with respect to the x^μ. A symmetry is *nonlocal* if the corresponding characteristic Q contains additional variables expressed as *integrals* of u with respect to the x^μ (or, more generally, integrals of local functions of u). As an example, the PSDYM characteristic $\Phi = [X, M]$ (where M is a constant matrix) represents a local symmetry of this PDE (since it depends locally on the PSDYM variable X), whereas the SDYM characteristic $Q = J[X, M]$ represents a nonlocal symmetry of that PDE since it contains an additional variable X which is expressed as an integral of a local function of the principal SDYM variable J [this follows from the BT (2)]. The infinite symmetries (34), (36) and (37) are increasingly nonlocal in X for $n > 0$, since they are produced by repeated application of the integro-differential recursion operator $\hat{R}$.

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Symmetry and Integrability of a Reduced, 3-Dimensional Self-Dual Gauge Field Model

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Abstract: A 3-dimensional reduction of the self-dual Yang-Mills (SDYM) equation, named SDYM3, is examined from the point of view of its symmetry and integrability characteristics. By using a non-auto-Bäcklund transformation, this equation is connected to its potential form (PSDYM3) and a certain isomorphism between the Lie algebras of symmetries of the two systems is shown to exist. This isomorphism allows us to study the infinite-dimensional Lie algebraic structure of the "potential symmetries" of SDYM3 by examining the symmetry structure of PSDYM3 (which is an easier task). By using techniques described in a recent paper, the recursion operators for both SDYM3 and PSDYM3 are derived. Moreover, a Lax pair and an infinite set of nonlocal conservation laws for SDYM3 are found, reflecting the fact that SDYM3 is a totally integrable system. This system may physically represent gravitational fields or chiral fields.

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1. Introduction

In a recent paper [1] we proposed a scheme by which symmetry and integrability aspects of a certain class of nonlinear partial differential equations (PDEs) are interrelated. We showed how, by starting with the symmetry condition of a PDE, one may derive integrability characteristics such as a Lax pair and an infinite set of (typically nonlocal) conservation laws. Moreover, we described an algorithm for constructing a recursion operator which, in principle, produces an infinite number of symmetries of the PDE from any given one.

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As examples, we applied these ideas to two systems of physical interest: the two-dimensional chiral field equation and the full, 4-dimensional self-dual Yang-Mills (SDYM) equation. The former system is a 2-dimensional reduction of the latter, thus shares some of its properties. However, there are differences: Although the SDYM recursion operator produces infinite sets of nontrivial symmetries when acting on both internal and coordinate symmetry transformations [2, 3], the chiral-field recursion operator yields infinite sets of internal symmetries only.

In this paper, we study an intermediate model which represents a 3-dimensional reduction of SDYM. We will name it SDYM3. With appropriate adjustments, this model may describe physical systems such as the complexified Ernst equation [4] or the 3-dimensional chiral field equation [5]. Happily, some important symmetry properties of SDYM, which are absent in the 2-dimensional chiral-field model, are restored in the 3-dimensional case. Thus, the SDYM3 model possesses infinite sets of nontrivial symmetries on both the base space (coordinate symmetries) and the fiber space (internal symmetries).

The Lie algebraic structure of symmetries of SDYM3 is certainly of interest. Although this aspect of the problem will be treated in full in a subsequent paper, some basic ideas are presented here. In the spirit of a recent paper on SDYM [6], we employ the concept of a Bäcklund transformation (BT) to connect SDYM3 with its counterpart in potential form, to be called PSDYM3. This BT also allows one to connect symmetries and recursion operators of the two systems. In particular, the symmetries of PSDYM3 yield “potential symmetries” [6-8] of SDYM3. It is proven that a Lie algebra isomorphism exists between the symmetries of SDYM3 and those of PSDYM3. Thus, to determine the Lie algebraic structure of symmetries of the former system, it suffices to study the corresponding structure of the latter system. This is not just a matter of academic significance but is important for practical reasons also, given that, as will be seen, the PSDYM3 recursion operator is simpler in form compared to the corresponding SDYM3 operator, with the result that the various commutation relations are easier to handle in the PSDYM3 case.

The paper is organized as follows:

In Section 2, the SDYM3-PSDYM3 system and its symmetry conditions are presented.

In Sec.3, the SDYM3 recursion operator is found in the form of a BT for the linear symmetry condition. This operator produces, in principle, an infinite number of symmetry characteristics, which are also seen to be conserved “charges” for SDYM3. A Lax pair for this PDE is also found.

In Sec.4, it is shown that the symmetries of PSDYM3 can be used to construct potential symmetries for SDYM3.

In Sec.5, a Lie algebra isomorphism is shown to exist between the symmetries of SDYM3 and those of PSDYM3. The practical usefulness of this isomorphism is explained.

The concept of isomorphically related (equivalent) recursion operators [6] is introduced in Sec.6. It is proven that the SDYM3 and PSDYM3 recursion operators are equivalent, thus they produce isomorphic symmetry subalgebras for the respective PDEs.

Finally, in Sec.7 we study the existence of infinite-dimensional abelian subalgebras of symmetries of PSDYM3, thus also of SDYM3. The presence of such algebras is a typical characteristic of integrable systems.

To facilitate the reader, we include an Appendix which contains definitions of the key concepts of the total derivative and the Fréchet derivative. For fuller and more rigorous definitions, the reader is referred to the book by Olver [9].

2. The SDYM3 – PSDYM3 System

We write the SDYM3 equation in the form

$$F\,[J] \;\equiv D_{\bar{y}}(J^{-1}J_y) + D_z(J^{-1}J_z) \;=\; 0 \tag{1}$$

(the bracket notation is explained in the Appendix). We denote by $x^\mu \equiv y,\ z,\ \bar{y}\quad(\mu = 1,\,2,\,3)$ the independent variables (assumed complex) and by $D_y\,,\,D_z\,,\,D_{\bar{y}}$ the total derivatives with respect to these variables. These derivatives will also be denoted by using subscripts [a mixed notation appears in Eq.(1)]. We assume that J is $SL(N,C)$-valued (i.e., $\det J = 1$).

We consider the non-auto-BT

$$J^{-1}J_y = X_z\ ,\quad J^{-1}J_z = -X_{\bar{y}} \tag{2}$$

The integrability condition $(X_{\bar{y}})_z = (X_z)_{\bar{y}}$ yields the SDYM3 equation (1). The integrability condition $(J_y)_z = (J_z)_y$, which is equivalent to

$$D_y\,(J^{-1}J_z) - D_z\,(J^{-1}J_y) \;+\; [J^{-1}J_y\,,\ J^{-1}J_z] \;=\; 0,$$

yields a nonlinear PDE for the "potential" X of Eq.(1), called the "potential SDYM3 equation" or PSDYM3:

$$G\,[X] \;\equiv\; X_{y\bar{y}} + X_{zz} + \;[\,X_z\,,\,X_{\bar{y}}] \;=\; 0 \tag{3}$$

Noting that, according to Eq.(2), $(trX)_z = [tr\,(\ln J)]_y = [\ln\,(\det J)]_y$, etc., we see that the condition $\det J = 1$ can be satisfied by requiring that trX=0 [this requirement is compatible with the PSDYM3 equation (3)]. Hence, $SL(N,C)$ SDYM3 solutions correspond to $sl(N,C)$ PSDYM3 solutions.

At this point we introduce the covariant derivative operators

$$\hat{A}_y \;\equiv\; D_y + \;[J^{-1}J_y\,,\ \;] \;=\; D_y + \;[X_z\,,\ \;]$$
$$\hat{A}_z \;\equiv\; D_z + \;[J^{-1}J_z\,,\ \;] \;=\; D_z - \;[X_{\bar{y}}\,,\ \;]$$

where the BT (2) has been taken into account. By using Eq.(3) and the Jacobi identity, the zero-curvature condition $[\hat{A}_y\,,\,\hat{A}_z] = 0$ is shown to be satisfied, as expected in view of the fact that the "connections" $J^{-1}J_y$ and $J^{-1}J_z$ are pure gauges. Moreover, the above operators are derivations on the Lie algebra of $sl(N,C)$-valued functions, satisfying a Leibniz rule of the form

$$\hat{A}_y\,[M,\,N] \;=\; [\hat{A}_yM,\ N] + [M,\ \hat{A}_yN]$$
$$\hat{A}_z\,[M,\,N] \;=\; [\hat{A}_zM,\ N] + [M,\ \hat{A}_zN]$$

for any matrix functions M, N.

Let $\delta J = \alpha\, Q\,[J]$ and $\delta X = \alpha\, \Phi\,[X]$ be infinitesimal symmetries of Eqs.(1) and (3), respectively (α is an infinitesimal parameter), with corresponding symmetry characteristics Q and Φ. (We note that *any* symmetry of a PDE can be expressed as a transformation of the dependent variable alone [7, 9], i.e., is equivalent to a "vertical" symmetry.) We will denote by $\Delta\, M\,[J]$ the Fréchet derivative (see Appendix) of a function M with respect to the characteristic Q. Similarly, by $\Delta\, N\,[X]$ we will denote the Fréchet derivative of a function N with respect to Φ. In particular, $\Delta\, J = Q$ and $\Delta\, X = \Phi$. The symmetry conditions for the PDEs (1) and (3) are, respectively,

$\Delta\, F\,[J] = 0 \mod F\,[J]$ and $\Delta\, G\,[X] = 0 \mod G\,[X]$.

By using the commutativity of the Fréchet derivative with total derivatives (see Appendix), and the fact that

$$\Delta\,(J^{-1}J_y) \;=\; \hat{A}_y\,(J^{-1}Q)\;, \quad \Delta\,(J^{-1}J_z) \;=\; \hat{A}_z\,(J^{-1}Q) \quad ,$$

the first of the above conditions leads to a linear PDE for the characteristic Q:

$$S(Q\,;\,J) \;\equiv\; (D_{\bar{y}}\hat{A}_y + D_z\hat{A}_z)\,(J^{-1}Q) \;=\; 0 \mod F\,[J] \tag{4}$$

which represents the symmetry condition for SDYM3.

The symmetry condition for PSDYM3 reads:

$$(\hat{A}_y D_{\bar{y}} + \hat{A}_z D_z)\,\Phi \;=0 \quad .$$

By using the operator identity

$$\begin{aligned} \hat{A}_y D_{\bar{y}} + \hat{A}_z D_z &= D_{\bar{y}}\hat{A}_y + D_z\hat{A}_z - \quad [F\,[J]\;,\;\;] \\ &= D_{\bar{y}}\hat{A}_y + D_z\hat{A}_z \mod F\,[J] \end{aligned} \tag{5}$$

we get the linear PDE for Φ :

$$S\,(\Phi\,;\,X) \equiv (D_{\bar{y}}\hat{A}_y + D_z\hat{A}_z)\,\Phi = 0 \mod G\,[X] \tag{6}$$

By comparing Eqs.(4) and (6), we notice that $J^{-1}Q$ and Φ satisfy the same PDE. Hence, we conclude that

if Q is an SDYM3 symmetry characteristic, then $\Phi = J^{-1}Q$ is a PSDYM3 characteristic.

Conversely,

if Φ is a PSDYM3 symmetry characteristic, then $Q = J\Phi$ is an SDYM3 characteristic.

3. Recursion Operator, Conserved Charges, and Lax Pair

We seek a recursion operator [9] for SDYM3, i.e., a linear operator which produces new symmetry characteristics Q' from "old" ones, Q. As in [1], we want to express this operator in the form of an auto-BT for the linear PDE (4) (which represents the symmetry condition for SDYM3). Moreover, this transformation must be consistent with the physical requirement $tr\,(J^{-1}Q) = 0$ (i.e., Q' must satisfy this property if Q does), which is necessary in order that the $SL(N,C)$ character of the SDYM3 solution be preserved.

The auto-BT for the PDE (4) is similar to that found in [1] for SDYM. Specifically,

$$\hat{A}_y(J^{-1}Q) = (J^{-1}Q')_z \;, \quad \hat{A}_z(J^{-1}Q) = -(J^{-1}Q')_{\bar{y}} \tag{7}$$

Integrability for Q' requires that Q satisfy Eq.(4). Integrability for Q, expressed by the condition $[\hat{A}_y\,,\ \hat{A}_z]\,(J^{-1}Q) = 0$, and upon using the operator identity (5), leads us again to Eq.(4), this time for Q'. The BT (7) may be regarded as an *invertible recursion operator* for the SDYM3 equation. It can be re-expressed as

$$\begin{aligned} \hat{A}_y\left(J^{-1}Q^{(n)}\right) &= D_z\left(J^{-1}Q^{(n+1)}\right) \\ \hat{A}_z\left(J^{-1}Q^{(n)}\right) &= -D_{\bar{y}}\left(J^{-1}Q^{(n+1)}\right) \end{aligned} \tag{8}$$

(n =0,±1,±2,...). From this we get a doubly infinite set of *nonlocal conservation laws* of the form

$$(D_{\bar{y}}\hat{A}_y + D_z\hat{A}_z)\left(J^{-1}Q^{(n)}\right) = 0 \mod F\,[J] \tag{9}$$

where the "conserved charges" $Q^{(n)}$ are symmetry characteristics.

Finally, the *Lax pair* for SDYM3, analogous to that found in [1] for SDYM, is

$$D_z\,(J^{-1}\Psi) = \lambda\,\hat{A}_y\,(J^{-1}\Psi) \;, \quad D_{\bar{y}}\,(J^{-1}\Psi) = -\lambda\,\hat{A}_z\,(J^{-1}\Psi) \tag{10}$$

(where λ is a complex "spectral" parameter). The proof of the Lax-pair property is sketched as follows: By the integrability condition $(J^{-1}\Psi)_{z\bar{y}} - (J^{-1}\Psi)_{\bar{y}z} = 0$, we get:

$$S(\Psi\,;\,J) \equiv (D_{\bar{y}}\hat{A}_y + D_z\hat{A}_z)\,(J^{-1}\Psi) = 0 \quad .$$

On the other hand, the integrability condition $[\hat{A}_y\,,\ \hat{A}_z]\,(J^{-1}\Psi) = 0$, by using the operator identity (5), yields:

$$S\,(\Psi\,;\,J) - \left[F\,[J]\,,\ J^{-1}\Psi\right] = 0 \quad .$$

Therefore, $[F\,[J]\,,\ J^{-1}\Psi] = 0$. This is valid independently of Ψ if $F[J]$=0, i.e., if J is an SDYM3 solution. We conclude that the linear system (10) is a Lax pair for the SDYM3 equation (1), the solution Ψ of which pair is a symmetry characteristic satisfying Eq.(4): $S\,(\Psi\,;\,J) = 0$. This Lax pair is different from that found by Nakamura for the Ernst equation [4].

4. Potential Symmetries of SDYM3

We recall that every SDYM3 symmetry characteristic can be expressed as Q=$J\Phi$, where Φ is a PSDYM3 characteristic. Let Φ be a characteristic which depends locally or nonlocally on X and/or various derivatives of X. By the BT (2), X must be an integral of J and its derivatives, and so it and its derivative X_y are nonlocal in J. On the other hand, according to Eq.(2), the quantities $X_{\bar{y}}$ and X_z depend locally on J. Thus, in general, Φ can be local or nonlocal in J. In the case where Φ is *nonlocal* in J, we say that the characteristic $Q{=}J\Phi$ expresses a *potential symmetry* of SDYM3 [7, 8].

Clearly, to obtain the complete set of potential symmetries of SDYM3, one must first find the totality of symmetries of PSDYM3. To this end, we need the recursion operator for the latter system. Having found the analogous operator for SDYM3, expressed by the BT (7), and by using the fact that $\Phi = J^{-1}Q$ is a PSDYM3 symmetry characteristic when Q is an SDYM3 characteristic, we easily get the recursion operator for PSDYM3 in the form of a BT for the symmetry condition (6):

$$\hat{A}_y \Phi = \Phi'_z \ , \quad \hat{A}_z \Phi = -\Phi'_{\bar{y}} \tag{11}$$

Since the above two PDEs are consistent with each other, we can use the first one to write $\Phi' = \hat{R}\,\Phi$, where we have introduced the linear operator

$$\hat{R} = D_z^{-1} \hat{A}_y \tag{12}$$

To show that the operator (12) is indeed a recursion operator for PSDYM3, we consider a symmetry characteristic Φ of Eq.(3), i.e., a solution of Eq.(6): $S(\Phi\,; X) = 0$. Then, by using the operator identity (5), and by taking into account the commutativity of covariant derivatives (zero-curvature condition), we have:

$$\begin{aligned} S(\Phi'; X) = S(\hat{R}\,\Phi\,; X) &= (\hat{A}_y D_{\bar{y}} + \hat{A}_z D_z)\,\hat{R}\,\Phi \\ &= \hat{A}_y D_z^{-1} D_{\bar{y}} \hat{A}_y \Phi + \hat{A}_z \hat{A}_y \Phi \\ &= \hat{A}_y D_z^{-1} (D_{\bar{y}} \hat{A}_y + D_z \hat{A}_z)\,\Phi + [\hat{A}_z\,,\ \hat{A}_y]\,\Phi \\ &= \hat{A}_y D_z^{-1} S(\Phi\,; X) + [\hat{A}_z\,,\ \hat{A}_y]\,\Phi = 0 \end{aligned}$$

which proves that $\Phi' = \hat{R}\,\Phi$ is a symmetry when Φ is a symmetry.

For $sl(N,C)$ PSDYM3 solutions, the symmetry characteristic Φ must be traceless. Then, so will be the characteristic $\Phi' = \hat{R}\,\Phi$. That is, the recursion operator (12) preserves the $sl(N,C)$ character of PSDYM3.

As is easy to see, any power $\hat{R}^n$ $(n = 0,\ \pm 1,\ \pm 2, \cdots)$ of an invertible recursion operator also is a recursion operator. Thus, given any symmetry characteristic $\Phi^{(0)}$, one may obtain, in principle, an infinite set of characteristics:

$$\Phi^{(n)} = \hat{R}\,\Phi^{(n-1)} = \hat{R}^n\,\Phi^{(0)} \quad (n = 0,\ \pm 1,\ \pm 2, \cdots) \tag{13}$$

Let us see some examples of using the recursion operator (12) to find PSDYM3 symmetries and corresponding SDYM3 potential symmetries:

1. Take $\Phi^{(0)} = M$, where M is a constant, traceless matrix. Then,

$$\Phi^{(1)} = \hat{R}\,\Phi^{(0)} = [X,\ M] \quad .$$

The corresponding potential symmetry of SDYM3 is

$$Q^{(1)} = J\Phi^{(1)} = J\,[X,\, M] \quad .$$

(This is nonlocal in J due to the presence of X.) Higher-order potential symmetries are found recursively by repeated application of the recursion operator (12). We thus obtain an infinite sequence of "internal" symmetries (i.e., symmetry transformations in the "fiber" space), of the form:

$$Q^{(n)} = J\Phi^{(n)} = J\hat{R}^n\Phi^{(0)} \quad (n = 0,\, 1,\, 2,\, \cdots) \tag{14}$$

In the case of the complexified Ernst equation, these are precisely the internal symmetries found by Nakamura [4].

2. Take $\Phi^{(0)} = X_y$, which represents a coordinate symmetry (symmetry transformation in the "base" space of the independent variables x^μ), specifically, invariance under $y-$translation. By applying the recursion operator, we get:

$$\Phi^{(1)} = \hat{R}\,\Phi^{(0)} = D_z^{-1}\,(X_{yy} + [X_z\,,\, X_y]\,) = D_z^{-1}\,(X_{yy} + [J^{-1}J_y\,,\, X_y]\,) \quad .$$

Both $\Phi^{(0)}$ and $\Phi^{(1)}$ are nonlocal in J (due to the presence of the $y-$derivatives of X, as well as of the integral operator with respect to z). We thus obtain the potential symmetries of SDYM3,

$$\begin{aligned} Q^{(0)} &= J\,\Phi^{(0)} = JX_y\ , \\ Q^{(1)} &= J\,\Phi^{(1)} = JD_z^{-1}\,(X_{yy} + [J^{-1}J_y\,,\, X_y]\,)\ . \end{aligned}$$

We note that, by applying the recursion operator to the translational characteristics $\Phi^{(0)} = X_z$ and $\Phi^{(0)} = X_{\bar{y}}$ [both of which are *local* in J, in view of the BT (2)], we get, respectively, $\Phi^{(1)} = X_y$ (which is nothing new) and $\Phi^{(1)} = -X_z$ (again, nothing new).

3. Take $\Phi^{(0)} = yX_y + zX_z + \bar{y}X_{\bar{y}}$, which represents a scale change of the x^μ. This is nonlocal in J due to X_y. We leave it to the reader to show that

$$\Phi^{(1)} = \hat{R}\,\Phi^{(0)} = zX_y - \bar{y}J^{-1}J_y + yD_z^{-1}\,(X_{yy} + [J^{-1}J_y\,,\, X_y]\,) \quad ,$$

where the PSDYM3 equation (3) and the BT (2) have been taken into account. This is also nonlocal in J. We conclude that $J\Phi^{(0)}$ and $J\Phi^{(1)}$ are potential symmetries for SDYM3.

5. Lie Algebra Isomorphism

We now study the connection between the Lie algebras of symmetries of SDYM3 and PSDYM3. If these algebras are isomorphic, then any Lie algebraic conclusion regarding the PSDYM3 equation will also be true for the SDYM3 equation. What is the practical value of this? As we saw, the recursion operator for PSDYM3 is given by Eq.(12):

$$\hat{R} = D_z^{-1}\hat{A}_y \quad .$$

On the other hand, from the BT (7) we get, by using the first equation,

$$Q' = J D_z^{-1} \hat{A}_y (J^{-1}Q) = J \hat{R}\,(J^{-1}Q) \;\equiv\; \hat{T}\,Q$$

where $\hat{T}$ is the operator form of the SDYM3 recursion operator:

$$\hat{T} = \; J D_z^{-1} \hat{A}_y \, J^{-1} = \; J \hat{R}\, J^{-1} \tag{15}$$

Obviously, $\hat{R}$ is of a simpler form compared to $\hat{T}$. Accordingly, the Lie algebraic structure of the infinite sequences of symmetries generated by the former operator will be easier to study compared to the corresponding structure of symmetries produced by the latter operator. So, we are seeking an isomorphism between the Lie algebras of symmetries of the PDEs (1) and (3).

In the spirit of [6], where the 4-dimensional SDYM case was studied, we consider the pair of PDEs:

$$\hat{A}_y (J^{-1}Q) \;=\; \Phi_z \;, \qquad \hat{A}_z (J^{-1}Q) \;=\; -\Phi_{\bar{y}} \tag{16}$$

Equation (16) is a BT connecting the symmetry characteristic Φ of PSDYM3 with the symmetry characteristic Q of SDYM3. Indeed, the integrability condition $(\Phi_z)_{\bar{y}} = (\Phi_{\bar{y}})_z$ yields the symmetry condition (4) for SDYM3, while the integrability condition $[\hat{A}_y \,,\, \hat{A}_z]\,(J^{-1}Q) = \,0$, valid in view of the zero-curvature condition, yields the PSDYM3 symmetry condition (6). Please note carefully that the Q and Φ in Eq.(16) are *not* related by the simple algebraic relation $Q{=}J\Phi$. Note also that the system (16) is compatible with the constraints that Φ and $J^{-1}Q$ be traceless, as required for producing $sl(N,C)$ PSDYM3 solutions and $SL(N,C)$ SDYM3 solutions, respectively.

We observe that, for a given Q, the solution of the BT (16) for Φ is not unique, and neither is the solution for Q, for a given Φ. Indeed, in either case the solution may contain arbitrary additive terms. We normalize the process by agreeing to ignore such terms, so that, in particular, the characteristic Q =0 corresponds to the characteristic Φ=0. In this way, the BT (16) establishes a one-to-one correspondence between the symmetries of SDYM3 and those of PSDYM3. We will now show that this correspondence is a Lie algebra isomorphism.

Lemma: The Fréchet derivative Δ with respect to the characteristic Φ, and the recursion operator $\hat{R}$ of Eq.(12), satisfy the commutation relation

$$[\Delta \;,\, \hat{R}] \;=\; D_z^{-1}\,[\Phi_z \;,\;\;] \tag{17}$$

where $\Phi = \Delta X$.

Proof: Introducing an auxiliary matrix function M, and using the derivation property of Δ and the commutativity of Δ with all total derivatives (as well as all powers of such derivatives), we have:

$$\begin{aligned}
\Delta\, \hat{R} M \;&=\; \Delta\, D_z^{-1} \hat{A}_y M \;\;=\; D_z^{-1} \Delta\, (D_y M + \;[X_z \;,\, M]\,) \\
&= \; D_z^{-1}\,(D_y \Delta M + \;[(\Delta X)_z \;,\, M] \;+\; [X_z \;,\, \Delta M]\,) \\
&= \; D_z^{-1}\,(\hat{A}_y \Delta M + \;[\Phi_z \;,\, M]\,) \;\;=\; \hat{R}\,\Delta M + \; D_z^{-1}\,[\Phi_z \;,\, M] \;,
\end{aligned}$$

from which there follows (17).

Now, by the first equation of the BT (16), we can write:

$$\Phi \;=\; D_z^{-1}\hat{A}_y\,(J^{-1}Q) \;=\; \hat{R}\,(J^{-1}Q) \tag{18}$$

Equation (18) defines a linear map from the set of symmetries Q=ΔJ of the PDE (1) to the set of symmetries $\Phi = \Delta X$ of the PDE (3). With the normalization conventions mentioned earlier, this map can be considered invertible, thus constituting a one-to-one correspondence between the symmetries of SDYM3 and those of PSDYM3, for any given solutions J and X connected to each other by the BT (2). Calling this map I, we write:

$$I: \;\; \Phi = I\,\{Q\} \;=\; \hat{R}\,J^{-1}Q \quad or \quad \Delta\,X = I\,\{\Delta\,J\} \;=\; \hat{R}\,J^{-1}\Delta\,J \tag{19}$$

[*Note:* We may omit parentheses, such as those in Eq.(18), by agreeing that an operator acts on the entire expression (e.g., product of functions) on its right, not just on the function adjacent to it. Hence, $\hat{P}MN \equiv \hat{P}\,(MN)$.]

Proposition 1: The map I defined by Eq.(19) is an isomorphism between the symmetry Lie algebras of SDYM3 and PSDYM3.

Proof: Consider a pair of symmetries of Eq.(1), indexed by i and j, generated by the characteristics $Q^{(\,l\,)} = \Delta^{(\,l\,)}J$, where $l = i,\, j$. Similarly, consider a pair of symmetries of Eq.(3), generated by $\Phi^{(\,l\,)} = \Delta^{(\,l\,)}X$ $(l = i,\, j)$. Further, assume that

$$\Phi^{(\,l\,)} = I\,\{Q^{(\,l\,)}\} \quad or \quad \Delta^{(\,l\,)}X = I\,\{\Delta^{(\,l\,)}J\,\} \quad .$$

That is,

$$\Phi^{(\,l\,)} = \Delta^{(\,l\,)}X = \;\hat{R}\,J^{-1}\Delta^{(\,l\,)}J = \;\hat{R}\,J^{-1}Q^{(\,l\,)}\;; \quad l = i,\, j \tag{20}$$

By the Lie-algebraic property of symmetries of PDEs, the functions $[\Delta^{(i)},\ \Delta^{(j)}]\,J$ and $[\Delta^{(i)},\ \Delta^{(j)}]\,X$ also are symmetry characteristics for SDYM3 and PSDYM3, respectively, where we have put

$$[\Delta^{(i)},\ \Delta^{(j)}]\,J \equiv\; \Delta^{(i)}\Delta^{(j)}J -\; \Delta^{(j)}\Delta^{(i)}J =\; \Delta^{(i)}Q^{(j)} -\; \Delta^{(j)}Q^{(i)}\,,$$

$$[\Delta^{(i)},\ \Delta^{(j)}]\,X \equiv\; \Delta^{(i)}\Delta^{(j)}X -\; \Delta^{(j)}\Delta^{(i)}X =\; \Delta^{(i)}\Phi^{(j)} -\; \Delta^{(j)}\Phi^{(i)}\,.$$

We must now show that

$$[\Delta^{(i)},\ \Delta^{(j)}]\,X = \; I\;\{[\Delta^{(i)},\ \Delta^{(j)}]\;J\,\} \;=\; \hat{R}\,J^{-1}\,[\Delta^{(i)},\ \Delta^{(j)}]\;J \tag{21}$$

Putting $l = j$ into Eq.(20), and applying the Fréchet derivative $\Delta^{(i)}$, we have:

$$\begin{aligned}\Delta^{(i)}\Delta^{(j)}X =\; \Delta^{(i)}\hat{R}\,J^{-1}Q^{(j)} &=\; [\Delta^{(i)},\ \hat{R}]\,J^{-1}Q^{(j)} +\; \hat{R}\,\Delta^{(i)}J^{-1}Q^{(j)} \\ &=\; D_z^{-1}\,[\Phi_z^{(i)},\ J^{-1}Q^{(j)}] \;+\; \hat{R}\,\Delta^{(i)}J^{-1}Q^{(j)}\,,\end{aligned}$$

where we have used the commutation relation (17). By Eq.(20),

$$\Phi_z^{(i)} =\; D_z\hat{R}\,J^{-1}Q^{(i)} =\; \hat{A}_y\,J^{-1}Q^{(i)}.$$

Moreover, by the properties of the Fréchet derivative listed in the Appendix,

$$\begin{aligned}\Delta^{(i)} J^{-1} Q^{(j)} &= -J^{-1} \left(\Delta^{(i)} J\right) J^{-1} Q^{(j)} + J^{-1} \Delta^{(i)} Q^{(j)} \\ &= -J^{-1} Q^{(i)} J^{-1} Q^{(j)} + J^{-1} \Delta^{(i)} Q^{(j)} .\end{aligned}$$

So,

$$\Delta^{(i)} \Delta^{(j)} X = D_z^{-1} [\hat{A}_y J^{-1} Q^{(i)}, \ J^{-1} Q^{(j)}] - \hat{R} J^{-1} Q^{(i)} J^{-1} Q^{(j)} + \hat{R} J^{-1} \Delta^{(i)} Q^{(j)}.$$

Subtracting from this the analogous expression for $\Delta^{(j)} \Delta^{(i)} X$, we have:

$$\begin{aligned}[\Delta^{(i)}, \ \Delta^{(j)}] X \equiv\ & \Delta^{(i)} \Delta^{(j)} X - \Delta^{(j)} \Delta^{(i)} X \\ =\ & D_z^{-1} \left([\hat{A}_y J^{-1} Q^{(i)}, \ J^{-1} Q^{(j)}] + [J^{-1} Q^{(i)}, \ \hat{A}_y J^{-1} Q^{(j)}] \right) \\ & - \hat{R} [J^{-1} Q^{(i)}, \ J^{-1} Q^{(j)}] + \hat{R} J^{-1} (\Delta^{(i)} Q^{(j)} - \Delta^{(j)} Q^{(i)}) \\ =\ & D_z^{-1} \hat{A}_y [J^{-1} Q^{(i)}, \ J^{-1} Q^{(j)}] - \hat{R} [J^{-1} Q^{(i)}, \ J^{-1} Q^{(j)}] \\ & + \hat{R} J^{-1} (\Delta^{(i)} \Delta^{(j)} J - \Delta^{(j)} \Delta^{(i)} J) \\ =\ & \hat{R} J^{-1} [\Delta^{(i)}, \ \Delta^{(j)}] J\end{aligned}$$

where we have used the derivation property of $\hat{A}_y$. Thus, Eq.(21) has been proven.

6. Isomorphically Related Recursion Operators

Following [6], we now introduce the concept of *isomorphically related* (*I-related*) *recursion operators*. Let $\hat{S}$ be a recursion operator for the SDYM3 equation (1) [not necessarily that of Eq.(15)], and let $\hat{P}$ be a recursion operator for the PSDYM3 equation (3) [not necessarily that of Eq.(12)].

Definition: The linear operators $\hat{P}$ and $\hat{S}$ will be called *equivalent with respect to the isomorphism I* (or *I-equivalent*, or *I-related*) if the following condition is satisfied:

$$\hat{P} \Phi = I \{ \hat{S} Q \} \quad when \quad \Phi = I \{ Q \} \tag{22}$$

where Q and Φ are symmetry characteristics for the PDEs (1) and (3), respectively.

Proposition 2: Any $I-$related recursion operators $\hat{P}$ and $\hat{S}$ satisfy the following operator equation on the infinite-dimensional linear space of all SDYM3 symmetry characteristics:

$$\hat{P} \hat{R} J^{-1} = \hat{R} J^{-1} \hat{S} \tag{23}$$

where $\hat{R}$ is the operator defined in Eq.(12).

Proof: By Eqs.(19) and (22),

$$\hat{P} \Phi = \hat{R} J^{-1} \hat{S} Q \quad when \quad \Phi = \hat{R} J^{-1} Q \quad \Rightarrow \quad \hat{P} \hat{R} J^{-1} Q = \hat{R} J^{-1} \hat{S} Q$$

for all SDYM3 characteristics Q.

Proposition 3: The recursion operators $\hat{R}$ and $\hat{T}$, defined by Eqs.(12) and (15), are $I-$equivalent.

Proof: Simply note that the operator equation (23) is satisfied by putting $\hat{P} = \hat{R}$ and $\hat{S} = \hat{T}$, and by taking Eq.(15) into account.

Now, let $Q^{(0)}$ be some SDYM3 symmetry characteristic, and let $\Phi^{(0)}$ be the $I-$related PSDYM3 characteristic:

$$\Phi^{(0)} = I\{Q^{(0)}\} = \hat{R}\,J^{-1}Q^{(0)} \tag{24}$$

Consider also the infinite sets of symmetries of the PDEs (1) and (3), respectively:

$$Q^{(n)} = \hat{T}^n Q^{(0)}\ ;\quad n = 0,\, 1,\, 2,\, \cdots \tag{25}$$

$$\Phi^{(n)} = \hat{R}^n \Phi^{(0)}\ ;\quad n = 0,\, 1,\, 2,\, \cdots \tag{26}$$

Proposition 4: If the set (25) generates an infinite-dimensional Lie subalgebra of SDYM3 symmetries, then the set (26) generates an infinite-dimensional Lie subalgebra of PSDYM3 symmetries, isomorphic to the SDYM3 symmetry subalgebra.

Proof: Since the operators $\hat{R}$ and $\hat{T}$ are $I-$equivalent, by Eq.(24) we have:

$$\hat{R}\,\Phi^{(0)} = I\{\hat{T}\,Q^{(0)}\} \quad .$$

By iterating,

$$\hat{R}^n \Phi^{(0)} = I\{\hat{T}^n Q^{(0)}\} \quad or \quad \Phi^{(n)} = I\{Q^{(n)}\}\ ;\quad n = 0,\, 1,\, 2,\, \cdots \tag{27}$$

Call V and W the infinite-dimensional linear spaces spanned by the basis functions (25) and (26), respectively. The elements of V and W are, correspondingly, symmetry characteristics of SDYM3 and PSDYM3. Equation (27) defines an isomorphism between V and W. By assumption, the characteristics belonging to V generate a Lie subalgebra of the complete Lie algebra of symmetries of SDYM3. We must show that the elements of W generate an isomorphic subalgebra of PSDYM3 symmetries. To this end, consider two basis elements $Q^{(i)} = \Delta^{(i)}J$ and $Q^{(j)} = \Delta^{(j)}J$ of V. From these we construct the Lie bracket,

$$[\Delta^{(i)},\ \Delta^{(j)}]\,J \equiv \Delta^{(i)}\Delta^{(j)}J - \Delta^{(j)}\Delta^{(i)}J = \Delta^{(i)}Q^{(j)} - \Delta^{(j)}Q^{(i)}$$

which is an SDYM3 symmetry characteristic. This characteristic belongs to the subspace V (since this space generates a Lie algebra). Now, let

$$\Phi^{(l)} = \Delta^{(l)}X = I\{Q^{(l)}\}\ ;\quad l = i,\, j$$

be the basis elements of W which are $I-$related to the $Q^{(l)}$ $(l = i,\, j)$, in the way dictated by Eq.(27). By Eq.(21),

$$[\Delta^{(i)},\ \Delta^{(j)}]\,X \equiv \Delta^{(i)}\Phi^{(j)} - \Delta^{(j)}\Phi^{(i)} = I\{[\Delta^{(i)},\ \Delta^{(j)}]\,J\} \quad .$$

The quantity on the left is a PSDYM3 symmetry characteristic. Given that I is a map from V to W, this characteristic belongs to the subspace W. Thus, W is closed under the Lie bracket operation, which means that its elements generate a Lie subalgebra of PSDYM3 symmetries. This subalgebra is $I-$related, i.e. isomorphic, to the corresponding subalgebra of SDYM3 symmetries generated by the elements of V.

7. Infinite-Dimensional Abelian Subalgebras

The study of the complete symmetry Lie algebra of SDYM3 will be the subject of a future paper. Here, we confine ourselves to the existence of infinite-dimensional abelian subalgebras, the presence of which is a typical characteristic of integrable systems. Since the PDEs (1) and (3) constitute a 3-dimensional reduction of the 4-dimensional SDYM-PSDYM system, certain symmetry aspects of the latter system are expected to be present in the former one also. In particular, the PSDYM equation has been shown to possess Kac-Moody symmetry algebras associated with both internal and coordinate transformations [2]. These algebras possess infinite-dimensional abelian subalgebras. Such abelian structures exist for the reduced 3-dimensional system also. The following theorem follows directly from a more general one concerning the 4-dimensional PSDYM equation [2]:

Theorem: Consider a PSDYM3 symmetry, having a characteristic of the form

$$\Phi^{(0)} = \Delta^{(0)} X = \hat{L} X$$

where $\hat{L}$ is a linear operator. By repeated application of the recursion operator (12), we construct an infinite sequence of PSDYM3 characteristics,

$$\Phi^{(n)} = \Delta^{(n)} X = \hat{R}^n \Phi^{(0)} = \hat{R}^n \hat{L} X \ ; \quad n = 0,\, 1,\, 2,\, \cdots \tag{28}$$

We assume that the operator $\hat{L}$ obeys the commutation relations
$[\Delta^{(n)},\ \hat{L}] = 0$ and $[\hat{L}\ ,\ \hat{R}] \ = D_z^{-1}\,[D_z \hat{L} X\ ,\quad]$.

Then, the set (28) represents an infinite-dimensional abelian symmetry algebra:

$$[\Delta^{(m)},\ \Delta^{(n)}]\, X \equiv\ \Delta^{(m)} \Phi^{(n)} - \Delta^{(n)} \Phi^{(m)} =\ 0 \quad .$$

We note that the commutation relation (17) is written, in this case,

$$[\Delta^{(n)},\ \hat{R}] \ = D_z^{-1}\,[D_z \Delta^{(n)} X\ ,\quad] \ =\ D_z^{-1}\,[D_z \hat{R}^n \hat{L} X\ ,\quad] \quad .$$

As an example, it can be checked that the conditions of this theorem are satisfied for the linear operators $\hat{L}_1 = D_y$ and $\hat{L}_2 = y D_y + z D_z + \bar{y} D_{\bar{y}}$, corresponding to the PSDYM3 symmetries $\Phi^{(0)} = X_y$ and $\Phi^{(0)} = y X_y + z X_z + \bar{y} X_{\bar{y}}$, respectively. The $I-$related SDYM3 symmetries are $Q^{(0)} = J_y$ and $Q^{(0)} = y J_y + z J_z + \bar{y} J_{\bar{y}}$. We thus obtain two infinite-dimensional abelian subsymmetries of PSDYM3:

$$\Phi^{(n)} = \Delta^{(n)} X = \hat{R}^n X_y \ ; \quad n = 0,\, 1,\, 2,\, \cdots$$

$$\Phi^{(n)} = \Delta^{(n)} X = \hat{R}^n (\, y X_y + z X_z + \bar{y} X_{\bar{y}}) \ ; \quad n = 0,\, 1,\, 2,\, \cdots$$

and two $I-$related abelian (by Proposition 4) sybsymmetries of SDYM3:

$$Q^{(n)} = \Delta^{(n)} J = \hat{T}^n J_y \ ; \quad n = 0,\, 1,\, 2,\, \cdots$$

$$Q^{(n)} = \Delta^{(n)} J = \hat{T}^n (\, y J_y + z J_z + \bar{y} J_{\bar{y}}) \ ; \quad n = 0,\, 1,\, 2,\, \cdots$$

Summary

We have explored the symmetry and integrability characteristics of a 3-dimensional reduction of the full 4-dimensional self-dual Yang-Mills system. The former model is physically interesting since, with appropriate adjustments, it may describe chiral fields [5, 10] or axially-symmetric gravitational fields [4]. We have used the techniques described in [1] to derive a recursion operator, a Lax pair, and an infinite set of conserved "charges". We have studied the existence of potential symmetries, and we have investigated certain aspects of the Lie algebraic structure of symmetries of our model. The study of the full symmetry algebra of this model will be the subject of a future paper.

Appendix

To make this article as self-contained as possible, we define two key concepts that are being used, namely, the total derivative and the Fréchet derivative. The reader is referred to the extensive review article [11] by this author for more details. (It should be noted, however, that our present definition of the Fréchet derivative corresponds to the definition of the *Lie* derivative in that article. Since these two derivatives are *locally* indistinguishable, this discrepancy in terminology should not cause any concern mathematically.)

We consider the set of all PDEs of the form $F[u]=0$, where, for simplicity, the solutions u (which may be matrix-valued) are assumed to be functions of only two variables, x and t: $u=u(x,t)$. In general,

$$F[u] \equiv F(x,\ t,\ u,\ u_x,\ u_t,\ u_{xx},\ u_{tt},\ u_{xt},\ \cdots) \quad .$$

Geometrically, we say that the function F is defined in a *jet space* [9, 12] with coordinates x,t,u, and as many partial derivatives of u as needed for the given problem. A solution of the PDE $F[u]=0$ is then a surface in this jet space.

Let $F[u]$ be a given function in the jet space. When differentiating such a function with respect to x or t, both implicit (through u) and explicit dependence of F on these variables must be taken into account. If u is a scalar quantity, we define the *total derivative operators* D_x and D_t as follows:

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots$$
$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots$$

(note that the operators $\partial/\partial x$ and $\partial/\partial t$ concern only the explicit dependence of F on x and t). If, however, u is matrix-valued, the above representation has only symbolic significance and cannot be used for actual calculations. We must therefore define the total derivatives D_x and D_t in more general terms.

We define a linear operator D_x, acting on functions $F[u]$ in the jet space and having the following properties:

1. On functions $f(x,t)$ in the base space,

$$D_x f(x,t) = \partial f/\partial x \;\equiv\; \partial_x f.$$

2. On functions $F[u] = u$ or u_x, u_t, etc., in the "fiber" space,
$D_x u = u_x$, $D_x u_x = u_{xx}$, $D_x u_t = u_{tx} = u_{xt}$, etc.

3. The operator D_x is a derivation on the algebra of all functions $F[u]$ in the jet space (i.e., the Leibniz rule is satisfied):

$$D_x\,(F[u]\,G[u]) = (D_x F[u])\;G[u] \;+ F[u]\,D_x G[u] \quad .$$

We similarly define the operator D_t. Extension to higher-order total derivatives is obvious (although these latter derivatives are no longer derivations, i.e., they do not satisfy the Leibniz rule). The following notation has been used in this article:

$$D_x F[u] \equiv F_x[u] \;, \quad D_t F[u] \equiv F_t[u] \quad .$$

Finally, it can be shown that, for any matrix-valued functions A and B in the jet space, we have

$$(A^{-1})_x = -\,A^{-1} A_x A^{-1} \;\;, \quad (A^{-1})_t = -\,A^{-1} A_t A^{-1}$$

and

$$D_x\,[A\,,\,B] = \;[A_x\,,\,B] + [A\,,\,B_x] \;\;, \quad D_t\,[A\,,\,B] = \;[A_t\,,\,B] + [A\,,\,B_t]$$

where square brackets denote commutators.

Let now $\delta u \simeq \alpha\,Q\,[u]$ be an infinitesimal symmetry transformation (with characteristic $Q[u]$) for the PDE $F[u] = 0$. We define the *Fréchet derivative* with respect to the characteristic Q as a linear operator Δ acting on functions $F[u]$ in the jet space and having the following properties:

1. On functions $f(x,t)$ in the base space,

$$\Delta\, f(x,t) = 0$$

(this is a consequence of our liberty to choose all our symmetries to be in "vertical" form [7, 9]).

2. On $F[u] = u$,

$$\Delta\, u = Q\,[u] \quad .$$

3. The operator Δ commutes with total derivative operators of any order.

4. The Leibniz rule is satisfied:

$$\Delta\,(F[u]\,G[u]) = (\Delta\,F[u])\;G[u] \;+ F[u]\,\Delta G[u] \quad .$$

The following properties can be proven:

$$\Delta\,u_x = (\Delta\,u)_x = Q_x[u] \;\;, \quad \Delta\,u_t = (\Delta\,u)_t = Q_t[u]$$

$$\Delta\,(A^{-1}) = -\,A^{-1}(\Delta\,A)\;A^{-1} \quad ; \quad \Delta\,[A\,,\,B] = \;[\Delta A\,,\,B] + \;[A\,,\,\Delta B]$$

where A and B are any matrix-valued functions in the jet space.

If the solution u of the PDE is a scalar function (thus so is the characteristic Q), the Fréchet derivative with respect to Q admits a differential-operator representation of the form

$$\Delta = Q\frac{\partial}{\partial u} + Q_x\frac{\partial}{\partial u_x} + Q_t\frac{\partial}{\partial u_t} + Q_{xx}\frac{\partial}{\partial u_{xx}} + Q_{tt}\frac{\partial}{\partial u_{tt}} + Q_{xt}\frac{\partial}{\partial u_{xt}} + \cdots$$

Such representations, however, are not valid for PDEs in matrix form. In these cases we must resort to the general definition of the Fréchet derivative given above.

Finally, by using the Fréchet derivative, the symmetry condition for a PDE $F[u] = 0$ can be expressed as follows [7, 9]:

$$\Delta F[u] = 0 \mod F[u] \quad .$$

This condition yields a linear PDE for the symmetry characteristic Q, of the form

$$S(Q\,;\,u) = 0 \mod F[u] \quad .$$

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Physics > Classical Physics

Foundations of Newtonian Dynamics: An Axiomatic Approach for the Thinking Student

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Despite its apparent simplicity, Newtonian Mechanics contains conceptual subtleties that may cause some confusion to the deep-thinking student. These subtleties concern fundamental issues such as, e.g., the number of independent laws needed to formulate the theory, or, the distinction between genuine physical laws and derivative theorems. This article attempts to clarify these issues for the benefit of the student by revisiting the foundations of Newtonian Dynamics and by proposing a rigorous axiomatic approach to the subject. This theoretical scheme is built upon two fundamental postulates, namely, conservation of momentum and superposition property for interactions. Newton's Laws, as well as all familiar theorems of Mechanics, are shown to follow from these basic principles.

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Foundations of Newtonian Dynamics: An Axiomatic Approach for the Thinking Student [1]

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Abstract. Despite its apparent simplicity, Newtonian mechanics contains conceptual subtleties that may cause some confusion to the deep-thinking student. These subtleties concern fundamental issues such as, e.g., the number of independent laws needed to formulate the theory, or, the distinction between genuine physical laws and derivative theorems. This article attempts to clarify these issues for the benefit of the student by revisiting the foundations of Newtonian dynamics and by proposing a rigorous axiomatic approach to the subject. This theoretical scheme is built upon two fundamental postulates, namely, conservation of momentum and superposition property for interactions. Newton's laws, as well as all familiar theorems of mechanics, are shown to follow from these basic principles.

1. Introduction

Teaching introductory mechanics can be a major challenge, especially in a class of students that are not willing to take anything for granted! The problem is that, even some of the most prestigious textbooks on the subject may leave the student with some degree of confusion, which manifests itself in questions like the following:

- Is the law of inertia (Newton's first law) a law of motion (of free bodies) or is it a statement of existence (of inertial reference frames)?
- Are the first two of Newton's laws independent of each other? It appears that the first law is redundant, being no more than a special case of the second law!
- Is the second law a true law or a definition (of force)?
- Is the third law more fundamental than conservation of momentum, or is it the other way around?
- Does the "parallelogram rule" for composition of forces follow trivially from Newton's laws, or is an additional, independent principle required?
- And, finally, what is the minimum number of *independent* laws needed in order to build a complete theoretical basis for mechanics?

In this article we describe an axiomatic approach to introductory mechanics that is both rigorous and pedagogical. It purports to clarify issues like the ones mentioned above, at an early stage of the learning process, thus aiding the student to acquire a deep understanding of the basic ideas of the theory. It is not the purpose of this article, of course, to present an outline of a complete course of mechanics! Rather, we will focus on the most fundamental concepts and principles, those that are taught at the early chapters of dynamics (we will not be concerned with kinematics, since this subject confines itself to a description of motion rather than investigating the physical laws governing this motion).

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[1] See Note at the end of the article.

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The axiomatic basis of our approach consists of two fundamental postulates, presented in Section 3. The first postulate (*P1*) embodies both the existence of *inertial reference frames* and the *conservation of momentum*, while the second one (*P2*) expresses a *superposition principle for interactions*. The *law of inertia* is deduced from *P1*.

In Sec. 4, the concept of *force* on a particle subject to interactions is defined (as in *Newton's second law*) and *P2* is used to show that a composite interaction of a particle with others is represented by a vector sum of forces. Then, *P1* and *P2* are used to derive the *action-reaction law*. Finally, a generalization to systems of particles subject to external interactions is made.

For completeness of presentation, certain derivative concepts such as angular momentum, work, kinetic energy, etc., are discussed in Sec. 5. To make the article self-contained, proofs of all theorems are included.

2. A critical look at Newton's theory

There have been several attempts to reexamine Newton's laws even since Newton's time. Probably the most important revision of Newton's ideas – and the one on which modern mechanics teaching is based – is that due to Ernst Mach (1838-1916) (for a beautiful discussion of Mach's ideas, see the classic article by H. A. Simon [1]). Our approach differs in several aspects from those of Mach and Simon, although all these approaches share common characteristics in spirit. (For a historical overview of the various viewpoints regarding the theoretical basis of classical mechanics, see, e.g., the first chapter of [2].)

The question of the *independence* of Newton's laws has troubled many generations of physicists. In particular, still on this day some authors assert that the first law (the law of inertia) is but a special case of the second law. The argument goes as follows:

> *"According to the second law, the acceleration of a particle is proportional to the total force acting on it. Now, in the case of a free particle the total force on it is zero. Thus, a free particle must not be accelerating, i.e., its velocity must be constant. But, this is precisely what the law of inertia says!"*

Where is the error in this line of reasoning? Answer: The error rests in regarding the acceleration as an absolute quantity independent of the observer that measures it. As we well know, this is not the case. In particular, the only observer *entitled* to conclude that a non-accelerating object is subject to no net force is an *inertial observer,* one who uses an *inertial frame of reference* for his/her measurements. It is precisely the law of inertia that *defines* inertial frames and *guarantees* their existence. So, without the first law, the second law becomes indeterminate, if not altogether wrong, since it would appear to be valid relative to any observer regardless of his/her state of motion. It may be said that the first law defines the "terrain" within which the second law acquires a meaning. Applying the latter law without taking the former one into account would be like trying to play soccer without possessing a soccer field!

The completeness of Newton's laws is another issue. Let us see a significant example: As is well known, the *principle of conservation of momentum* is a direct consequence of Newton's laws. This principle dictates that the total momentum of a system of particles is constant in time, relative to an inertial frame of reference, when the total external force on the system vanishes (in particular, this is true for an *isolated* system of particles, i.e., a system subject to no external forces). But, when proving this principle we take it for granted that the total force on each particle is the vector sum of all forces (both internal and external) acting on it. This is *not* something that follows trivially from Newton's laws, however! In fact, it was Daniel Bernoulli who first stated this *principle of superposition* after Newton's death. This means that classical Newtonian mechanics is built upon a total of *four* – rather than just three – basic laws.

The question now is: can we somehow "compactify" the axiomatic basis of Newtonian mechanics in order for it to consist of a smaller number of independent principles? At this point it is worth taking a closer look at the principle of conservation of momentum mentioned above. In particular, we note the following:

- For an isolated "system" consisting of a single particle, conservation of momentum reduces to the law of inertia (the momentum, thus also the velocity, of a free particle is constant relative to an inertial frame of reference).
- For an isolated system of two particles, conservation of momentum takes us back to the action-reaction law (Newton's third law).

Thus, starting with four fundamental laws (the three laws of Newton plus the law of superposition) we derived a new principle (conservation of momentum) that yields, as special cases, two of the laws we started with. The idea is then that, by taking *this* principle as our fundamental physical law, the number of independent laws necessary for building the theory would be reduced.

How about Newton's second law? We take the view, adopted by several authors including Mach himself (see, e.g., [1,3-7]) that this "law" should be interpreted as the *definition* of force in terms of the rate of change of momentum.

We thus end up with a theory built upon *two* fundamental principles, i.e., the conservation of momentum and the principle of superposition. In the following sections these ideas are presented in more detail.

3. The fundamental postulates and their consequences

We begin with some basic definitions.

Definition 1. A *frame of reference* (or *reference frame*) is a system of coordinates (or axes) used by an observer to measure physical quantities such as the position, the velocity, the acceleration, etc., of any particle in space. The position of the observer him/herself is assumed *fixed* relative to his/her own frame of reference.

Definition 2. An *isolated system of particles* is a system of particles subject only to their mutual interactions, i.e., subject to no *external* interactions. Any system of particles subject to external interactions that somehow cancel one another in order to make the system's motion identical to that of an isolated system will also be considered "isolated". In particular, an isolated system consisting of a single particle is called a *free particle*.

Our first fundamental postulate of mechanics is stated as follows:

Postulate 1. A class of frames of reference (*inertial frames*) exists such that, for any *isolated* system of particles, a vector equation of the following form is valid:

$$\sum_i m_i \vec{v}_i = \text{constant in time} \qquad (1)$$

where $\vec{v}_i$ is the velocity of the particle indexed by i ($i = 1, 2, \cdots$) and where m_i is a constant quantity associated with this particle, which quantity is independent of the number or the nature of interactions the particle is subject to.

We call m_i the *mass* and $\vec{p}_i = m_i \vec{v}_i$ the *momentum* of the *i*th particle. Also, we call

$$\vec{P} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i \qquad (2)$$

the *total momentum* of the system relative to the considered reference frame. Postulate 1, then, expresses the *principle of conservation of momentum*: the total momentum of an isolated system of particles, relative to an inertial reference frame, is constant in time. (The same is true, in particular, for a free particle.)

Corollary 1. A free particle moves with constant velocity (i.e., with no acceleration) relative to an *inertial* reference frame.

Corollary 2. Any two free particles move with constant velocities relative to each other (their relative velocity is constant and their relative acceleration is zero).

Corollary 3. The position of a free particle may define the origin of an inertial frame of reference.

We note that Corollaries 1 and 2 constitute alternate expressions of the *law of inertia* (*Newton's first law*).

By *inertial observer* we mean an "intelligent" free particle, i.e., one that can perform measurements of physical quantities such as velocity or acceleration. By convention, the observer is assumed to be located at the origin of his/her own inertial frame of reference.

Corollary 4. Inertial observers move with constant velocities (i.e., they do not accelerate) relative to one another.

Consider now an isolated system of two particles of masses m_1 and m_2. Assume that the particles are allowed to interact for some time interval Δt. By conservation of momentum relative to an inertial frame of reference, we have:

$$\Delta(\vec{p}_1 + \vec{p}_2) = 0 \;\Rightarrow\; \Delta\vec{p}_1 = -\Delta\vec{p}_2 \;\Rightarrow\; m_1 \Delta\vec{v}_1 = -m_2 \Delta\vec{v}_2 \;.$$

We note that the changes in the velocities of the two particles within the (arbitrary) time interval Δt must be in opposite directions, a fact that is verified experimentally. Moreover, these changes are independent of the particular inertial frame used to measure the velocities (although, of course, the velocities themselves *are* frame-dependent!). This latter statement is a consequence of the constancy of the relative velocity of any two inertial observers (the student is invited to explain this in detail). Now, taking magnitudes in the above vector equation, we have:

$$\frac{|\Delta\vec{v}_1|}{|\Delta\vec{v}_2|} = \frac{m_2}{m_1} = \text{constant} \tag{3}$$

regardless of the kind of interaction or the time Δt (which also is an experimentally verified fact). These demonstrate, in practice, the validity of the first postulate. Equation (3) allows us to specify the mass of a particle numerically, relative to the mass of some other particle (which particle may arbitrarily be assigned a unit mass), by letting the two particles interact for some time. As argued above, the result will be independent of the specific inertial frame used by the observer who makes the measurements. That is, in the classical theory, *mass is a frame-independent quantity*.

So far we have examined the case of isolated systems and, in particular, free particles. Consider now a particle subject to interactions with the rest of the world. Then, in general (unless these interactions somehow cancel one another), the particle's momentum will not remain constant relative to an *inertial* reference frame, i.e., will be a function of time. Our second postulate, which expresses the *superposition principle for interactions*, asserts that external interactions act on a particle *independently of one another* and their effects are superimposed.

Postulate 2. If a particle of mass m is subject to interactions with particles $m_1, m_2, \cdots$, then, at each instant t, the rate of change of this particle's momentum relative to an inertial reference frame is equal to

$$\frac{d\vec{p}}{dt} = \sum_i \left(\frac{d\vec{p}}{dt}\right)_i \tag{4}$$

where $(d\vec{p}/dt)_i$ is the rate of change of the particle's momentum due solely to the interaction of this particle with the particle m_i (i.e., the rate of change of $\vec{p}$ *if* the particle m interacted *only* with m_i).

4. The concept of force and the Third Law

We now *define* the concept of force, in a manner similar to *Newton's second law*:

Definition 3. Consider a particle of mass m that is subject to interactions. Let $\vec{p}(t)$ be the particle's momentum as a function of time, as measured relative to an *inertial* reference frame. The vector quantity

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{5}$$

is called the *total force* acting on the particle at time t.

Taking into account that, for a single particle, $\vec{p} = m\vec{v}$ with fixed *m*, we may re-write Eq. (5) in the equivalent form,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \tag{6}$$

where $\vec{a}$ is the particle's acceleration at time *t*. Given that both the mass and the acceleration (prove this!) are independent of the inertial frame used to measure them, we conclude that *the total force on a particle is a frame-independent quantity*.

Corollary 5. Consider a particle of mass *m* subject to interactions with particles $m_1, m_2, \cdots$. Let $\vec{F}$ be the total force on *m* at time *t*, and let $\vec{F}_i$ be the force on *m* due solely to its interaction with m_i. Then, by the superposition principle for interactions (Postulate 2) as expressed by Eq. (4), we have:

$$\vec{F} = \sum_i \vec{F}_i \tag{7}$$

Theorem 1. Consider two particles *1* and *2*. Let $\vec{F}_{12}$ be the force on particle *1* due to its interaction with particle *2* at time *t*, and let $\vec{F}_{21}$ be the force on particle *2* due to its interaction with particle *1* at the same instant. Then,

$$\vec{F}_{12} = -\vec{F}_{21} \tag{8}$$

Proof. By the independence of interactions, as expressed by the superposition principle, the forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are independent of the presence or not of other particles in interaction with particles *1* and *2*. Thus, without loss of generality, we may assume that the system of the two particles is isolated. Then, by conservation of momentum and by using Eq. (5),

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0 \;\Rightarrow\; \frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt} \;\Rightarrow\; \vec{F}_{12} = -\vec{F}_{21} \;.$$

Equation (8) expresses the *action-reaction law* (*Newton's third law*).

Theorem 2. The rate of change of the total momentum $\vec{P}(t)$ of a system of particles, relative to an inertial frame of reference, equals the total *external* force acting on the system at time *t*.

Proof. Consider a system of particles of masses $m_i \; (i = 1, 2, \cdots)$. Let $\vec{F}_i$ be the total *external* force on m_i (due to its interactions with particles *not belonging* to the system), and let $\vec{F}_{ij}$ be the *internal* force on m_i due to its interaction with m_j (by convention, $\vec{F}_{ij} = 0$ when *i*=*j*). Then, by Eq. (5) and by taking into account Eq. (7),

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i + \sum_j \vec{F}_{ij} \ .$$

By using Eq. (2) for the total momentum, we have:

$$\frac{d\vec{P}}{dt} = \sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_i + \sum_{ij} \vec{F}_{ij} \ .$$

But,

$$\sum_{ij} \vec{F}_{ij} = \sum_{ji} \vec{F}_{ji} = \frac{1}{2} \sum_{ij} \left(\vec{F}_{ij} + \vec{F}_{ji} \right) = 0 \, ,$$

where the action-reaction law (8) has been taken into account. So, finally,

$$\frac{d\vec{P}}{dt} = \sum_i \vec{F}_i = \vec{F}_{ext} \tag{9}$$

where $\vec{F}_{ext}$ represents the *total external force* on the system.

5. Derivative concepts and theorems

Having presented the most fundamental concepts of mechanics, we now turn to some useful derivative concepts and related theorems, such as those of angular momentum and its relation to torque, work and its relation to kinetic energy, and conservative force fields and their association with mechanical-energy conservation.

Definition 4. Let *O* be the origin of an *inertial* reference frame, and let $\vec{r}$ be the position vector of a particle of mass *m*, relative to *O*. The vector quantity

$$\vec{L} = \vec{r} \times \vec{p} = m \left(\vec{r} \times \vec{v} \right) \tag{10}$$

(where $\vec{p} = m\vec{v}$ is the particle's momentum in the considered frame) is called the *angular momentum* of the particle relative to *O*.

Theorem 3. The rate of change of the angular momentum of a particle, relative to *O*, is given by

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \equiv \vec{T} \tag{11}$$

where $\vec{F}$ is the *total* force on the particle at time *t* and where $\vec{T}$ is the *torque* of this force relative to *O*, at this instant.

Proof. Equation (11) is easily proven by differentiating Eq. (10) with respect to time and by using Eq. (5).

Corollary 6. If the torque of the total force on a particle, relative to some point *O*, vanishes, then the angular momentum of the particle relative to *O* is constant in time (*principle of conservation of angular momentum*).

Under appropriate conditions, the above conservation principle can be extended to the more general case of a system of particles (see, e.g., [2-8]).

Definition 5. Consider a particle of mass *m* in a *force field* $\vec{F}(\vec{r})$, where $\vec{r}$ is the particle's position vector relative to the origin *O* of an inertial reference frame. Let *C* be a curve representing the trajectory of the particle from point *A* to point *B* in this field. Then, the line integral

$$W_{AB} = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} \tag{12}$$

represents the *work* done by the force field on *m* along the path *C*. (Note: This definition is valid independently of whether or not additional forces, not related to the field, are acting on the particle; i.e., regardless of whether or not $\vec{F}(\vec{r})$ represents the total force on *m*.)

Theorem 4. Let $\vec{F}(\vec{r})$ represent the *total* force on a particle of mass *m* in a force field. Then, the work done on the particle along a path *C* from *A* to *B* is equal to

$$W_{AB} = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} = E_{k,B} - E_{k,A} = \Delta E_k \tag{13}$$

where

$$E_k = \frac{1}{2} m v^2 = \frac{p^2}{2m} \tag{14}$$

is the *kinetic energy* of the particle.

Proof. By using Eq. (6), we have:

$$\vec{F} \cdot d\vec{r} = m \frac{d\vec{v}}{dt} \cdot d\vec{r} = m\vec{v} \cdot d\vec{v} = \frac{1}{2} m\, d(\vec{v} \cdot \vec{v}) = \frac{1}{2} m\, d(v^2) = m v\, dv \,,$$

from which Eq. (13) follows immediately.

Definition 6. A force field $\vec{F}(\vec{r})$ is said to be *conservative* if a scalar function $E_p(\vec{r})$ (*potential energy*) exists, such that the work on a particle along *any* path from *A* to *B* can be written as

$$W_{AB} = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} = E_{p,A} - E_{p,B} = -\Delta E_p \tag{15}$$

Theorem 5. If the total force $\vec{F}(\vec{r})$ acting on a particle m is conservative, with an associated potential energy $E_p(\vec{r})$, then the quantity

$$E = E_k + E_p = \frac{1}{2} m v^2 + E_p(\vec{r}) \tag{16}$$

(*total mechanical energy* of the particle) remains constant along any path traced by the particle (*conservation of mechanical energy*).

Proof. By combining Eq. (13) (which is generally valid for *any* kind of force) with Eq. (15) (which is valid for *conservative* force fields) we find:

$$\varDelta E_k = -\varDelta E_p \;\Rightarrow\; \varDelta\,(E_k + E_p) = 0 \;\Rightarrow\; E_k + E_p = const.$$

Theorems 4 and 5 are readily extended to the case of a system of particles (see, e.g., [2-8]).

6. Some conceptual problems

After establishing our axiomatic basis and demonstrating that the standard Newtonian laws are consistent with it, the development of the rest of mechanics follows familiar paths. Thus, as we saw in the previous section, we can define concepts such as angular momentum, work, kinetic and total mechanical energies, etc., and we can state derivative theorems such as conservation of angular momentum, conservation of mechanical energy, etc. Also, rigid bodies and continuous media can be treated in the usual way [2-8] as systems containing an arbitrarily large number of particles.

Despite the more "economical" axiomatic basis of Newtonian mechanics suggested here, however, certain problems inherent in the classical theory remain. Let us point out a few:

1. The problem of "inertial frames"

An inertial frame of reference is only a theoretical abstraction: such a frame cannot exist in reality. As follows from the discussion in Sec. 3, the origin (say, O) of an inertial frame coincides with the position of a hypothetical free particle and, moreover, any real free particle moves with constant velocity relative to O. However, no such thing as an absolutely free particle may exist in the world. In the first place, every material particle is subject to the infinitely long-range gravitational interaction with the rest of the world. Furthermore, in order for a supposedly inertial observer to measure the velocity of a "free" particle and verify that this particle is not accelerating relative to him/her, the observer must somehow interact with the particle. Thus, no matter how weak this interaction may be, the particle cannot be considered free in the course of the observation.

2. The problem of simultaneity

In Sec. 4 we used our two postulates, together with the definition of force, to derive the action-reaction law. Implicit in our arguments was the requirement that action must be *simultaneous* with reaction. As is well known, this hypothesis, which suggests instantaneous action at a distance, ignores the finite speed of propagation of the field associated with the interaction and violates causality.

3. A dimensionless "observer"

As we have used this concept, an "observer" is an intelligent free particle capable of making measurements of physical quantities such as velocity or acceleration. Such an observer may use any convenient (preferably rectangular) set of axes (x, y, z) for his/her measurements. Different systems of axes used by this observer have different orientations in space. By convention, the observer is located at the origin O of the chosen system of axes.

As we know, inertial observers do not accelerate relative to one another. Thus, the relative velocity of the origins (say, O and O') of two different inertial frames of reference is constant in time. But, what if the axes of these frames are in *relative rotation* (although the origins O and O' move uniformly relative to each other, or even coincide)? How can we tell which observer (if any) is an inertial one?

The answer is that, relative to the system of axes of an inertial frame, a free particle does not accelerate. In particular, relative to a rotating frame, a free particle will appear to possess at least a centripetal acceleration. Such a frame, therefore, cannot be inertial.

As mentioned previously, an object with finite dimensions (e.g., a rigid body) can be treated as an arbitrarily large system of particles. No additional postulates are thus needed in order to study the dynamics of such an object. This allows us to regard momentum and its conservation as more fundamental than angular momentum and its conservation, respectively. In this regard, our approach differs significantly from, e.g., that of Simon [1] who, in his own treatment, places the aforementioned two conservation laws on an equal footing from the outset.

7. Summary

Newtonian mechanics is the first subject in Physics an undergraduate student is exposed to. It continues to be important even at the intermediate and advanced levels, despite the predominant role played there by the more general formulations of Lagrangian and Hamiltonian dynamics.

It is this author's experience as a teacher that, despite its apparent simplicity, Newtonian mechanics contains certain conceptual subtleties that may leave the deep-thinking student with some degree of confusion. The average student, of course, is happy with the idea that the whole theory is built upon three rather simple laws attributed to Newton's genius. In the mind of the more demanding student, however, puzzling questions often arise, such as, e.g., how many independent laws we really need to fully formulate the theory, or, which ones should be regarded as truly fundamental laws of Nature, as opposed to others that can be derived as theorems.

This article suggested an axiomatic approach to introductory mechanics, based on two fundamental, empirically verifiable laws; namely, the *principle of conservation of momentum* and the *principle of superposition for interactions*. We showed that all standard ideas of mechanics (including, of course, Newton's laws) naturally follow from these basic principles. To make our formulation as economical as possible, we expressed the first principle in terms of a system of particles and treated the single-particle situation as a special case. To make the article self-contained for the benefit of the student, explicit proofs of all theorems were given.

By no means do we assert, of course, that this particular approach is unique or pedagogically superior to other established methods that adopt different viewpoints regarding the axiomatic basis of classical mechanics. Moreover, as noted in Sec. 6, this approach is not devoid of the usual theoretical problems inherent in Newtonian mechanics (see also [9,10]).

In any case, it looks like classical mechanics remains a subject open to discussion and re-interpretation, and more can always be said about things that are usually taken for granted by most students (this is not exclusively their fault, of course!). Happily, some of my own students do not fall into this category. I appreciate the hard time they enjoy giving me in class!

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[3] Available on the Internet; please search title.

Electromotive Force: A Guide for the Perplexed

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Abstract. The concept of electromotive force (emf) may be introduced in various ways in an undergraduate course of theoretical electromagnetism. The multitude of alternate expressions for the emf is often the source of confusion to the student. We summarize the main ideas, adopting a pedagogical logic that proceeds from the general to the specific. The emf of a "circuit" is first defined in the most general terms. The expressions for the emf of some familiar electrodynamical systems are then derived in a rather straightforward manner. A diversity of physical situations is thus unified within a common theoretical framework.

1. INTRODUCTION

The difficulty in writing this article was not just due to the subject itself: we had to first overcome some almost irreconcilable differences in educational philosophy between an (opinionated) theoretical physicist and an (equally -if not more- opinionated) electrical engineer. At long last, a compromise was reached! This paper is the fruit of this "mutual understanding".

Having taught intermediate-level electrodynamics courses for several years, we have come to realize that, in the minds of many of our students, the concept of *electromotive force* (*emf*) is something of a mystery. What is an emf, after all? Is it the voltage of an ideal battery in a DC circuit? Is it work per unit charge? Or is it, in a more sophisticated way, the line integral of the electric field along a closed path? And what if a magnetic rather than an electric field is present?

Generally speaking, the problem with the emf lies in the diversity of situations where this concept applies, leading to a multitude of corresponding expressions for the emf. The subject is discussed in detail, of course, in all standard textbooks on electromagnetism, both at the intermediate [1-9] and at the advanced [10-12] level. Here we summarize the main ideas, choosing a pedagogical approach that proceeds from the general to the specific. We begin by defining the concept of emf of a "circuit" in the most general way possible. We then apply this definition to certain electrodynamic systems in order to recover familiar expressions for the emf. The main advantage of this approach is that a number of different physical situations are treated in a unified way within a common theoretical framework.

The general definition of the emf is given in Section 2. In subsequent sections (Sec.3-5) application is made to particular cases, such as motional emf, the emf due to a time-varying magnetic field, and the emf of a DC circuit consisting of an ideal battery and a resistor. In Sec.6, the connection between the emf and Ohm's law is discussed.

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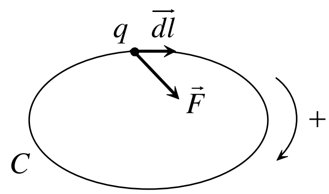
2. THE GENERAL DEFINITION OF EMF

Consider a region of space in which an electromagnetic (e/m) field exists. In the most general sense, any *closed* path *C* (or *loop*) within this region will be called a *"circuit"* (whether or not the whole or parts of *C* consist of material objects such as wires, resistors, capacitors, batteries, or any other elements whose presence may contribute to the e/m field).

We *arbitrarily* assign a positive direction of traversing the loop *C*, and we consider an element $\overrightarrow{dl}$ of *C* oriented in the positive direction. Imagine now a test charge *q* located at the position of $\overrightarrow{dl}$, and let $\vec{F}$ be the force on *q* at time *t*:



This force is exerted by the e/m field itself, as well as, possibly, by additional energy *sources* (e.g., batteries) that can interact electrically with *q*. The *force per unit charge* at the position of $\overrightarrow{dl}$ at time *t*, is

$$\vec{f} = \frac{\vec{F}}{q} \tag{1}$$

Note that $\vec{f}$ is independent of *q*, since the force by the e/m field and/or the sources on *q* is proportional to the charge. In particular, reversing the sign of *q* will have no effect on $\vec{f}$ (although it will change the direction of $\vec{F}$).

We now define the *electromotive force* (*emf*) of the circuit *C* at time *t* as the line integral of $\vec{f}$ along *C*, taken in the *positive* sense of *C*:

$$\mathcal{E} = \oint_C \vec{f} \cdot \overrightarrow{dl} \tag{2}$$

Note that the sign of the emf is dependent upon our choice of the positive direction of circulation of *C*: by changing this convention, the sign of $\mathcal{E}$ is reversed.

We remark that, in the *non-relativistic* limit, the emf of a circuit *C* is the same for all inertial observers since *at this limit* the force $\vec{F}$ is invariant under a change of frame of reference.

In the following sections we apply the defining equation (2) to a number of specific electrodynamic situations that are certainly familiar to the student.

3. MOTIONAL EMF IN THE PRESENCE OF A STATIC MAGNETIC FIELD

Consider a circuit consisting of a closed wire *C*. The wire is moving inside a *static* magnetic field $\vec{B}(\vec{r})$. Let $\vec{\upsilon}$ be the velocity of the element $\overrightarrow{dl}$ of *C* relative to our inertial frame of reference. A charge *q* (say, a free electron) at the location of $\overrightarrow{dl}$ executes a composite motion, due to the motion of the loop *C* itself relative to our frame, as well as the motion of *q along C*. The total velocity of *q* relative to us is $\vec{\upsilon}_{tot} = \vec{\upsilon} + \vec{\upsilon}'$, where $\vec{\upsilon}'$ is the velocity of *q* in a direction parallel to $\overrightarrow{dl}$. The force from the magnetic field on *q* is

$$\vec{F} = q\,(\vec{\upsilon}_{tot} \times \vec{B}) = q\,(\vec{\upsilon} \times \vec{B}) + q\,(\vec{\upsilon}' \times \vec{B}) \;\Rightarrow$$
$$\vec{f} = \frac{\vec{F}}{q} = (\vec{\upsilon} \times \vec{B}) + (\vec{\upsilon}' \times \vec{B})$$

By (2), then, the emf of the circuit *C* is

$$\mathcal{E} = \oint_C \vec{f} \cdot \overrightarrow{dl} = \oint_C (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} + \oint_C (\vec{\upsilon}' \times \vec{B}) \cdot \overrightarrow{dl}$$

But, since $\vec{\upsilon}'$ is parallel to $\overrightarrow{dl}$, we have that $(\vec{\upsilon}' \times \vec{B}) \cdot \overrightarrow{dl} = 0$. Thus, finally,

$$\mathcal{E} = \oint_C (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} \tag{3}$$

Note that the wire *need not maintain a fixed shape, size or orientation* during its motion! Note also that the velocity $\vec{\upsilon}$ may vary around the circuit.

By using (3), it can be proven (see Appendix) that

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{4}$$

where $\Phi = \int \vec{B} \cdot \overrightarrow{da}$ is the magnetic flux through the wire *C* at time *t*. Note carefully that (4) does not express any novel physical law: it is simply a direct consequence of the definition of the emf !

4. EMF DUE TO A TIME-VARYING MAGNETIC FIELD

Consider now a closed wire *C* that is *at rest* inside a *time-varying* magnetic field $\vec{B}(\vec{r},t)$. As experiments show, as soon as $\vec{B}$ starts changing, a current begins to flow in the wire. This looks impressive, given that the free charges in the (stationary) wire were initially at rest. And, as everybody knows, a magnetic field exerts forces on *moving* charges only! It is also observed experimentally that, if the magnetic field $\vec{B}$ stops varying in time, the current in the wire

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disappears. The only field that can put an initially stationary charge in motion and keep this charge moving is an *electric* field.

We are thus compelled to conclude that *a time-varying magnetic field is necessarily accompanied by an electric field.* (It is often said that "a changing magnetic field *induces* an electric field". This is somewhat misleading since it gives the impression that the "source" of an electric field could be a magnetic field. Let us keep in mind, however, that the true sources of any e/m field are the electric charges and the electric currents!)

So, let $\vec{E}(\vec{r},t)$ be the electric field accompanying the time-varying magnetic field $\vec{B}$. Consider again a charge *q* at the position of the element $\overrightarrow{dl}$ of the wire. Given that the wire is now at rest (relative to our inertial frame), the velocity of *q* will be due to the motion of the charge along the wire only, i.e., in a direction parallel to $\overrightarrow{dl}$: $\vec{\upsilon}_{tot} = \vec{\upsilon}'$ (since $\vec{\upsilon} = 0$). The force on *q* by the e/m field is

$$\vec{F} = q\,[\vec{E} + (\vec{\upsilon}_{tot} \times \vec{B})] = q\,[\vec{E} + (\vec{\upsilon}' \times \vec{B})] \;\Rightarrow$$
$$\vec{f} = \frac{\vec{F}}{q} = \vec{E} + (\vec{\upsilon}' \times \vec{B})$$

The emf of the circuit *C* is now

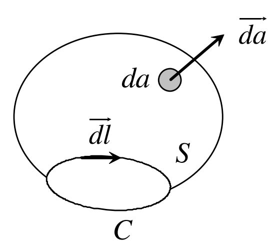
$$\mathcal{E} = \oint_C \vec{f} \cdot \overrightarrow{dl} = \oint_C \vec{E} \cdot \overrightarrow{dl} + \oint_C (\vec{\upsilon}' \times \vec{B}) \cdot \overrightarrow{dl}$$

But, as explained earlier, $(\vec{\upsilon}' \times \vec{B}) \cdot \overrightarrow{dl} = 0$. Thus, finally,

$$\mathcal{E} = \oint_C \vec{E} \cdot \overrightarrow{dl} \qquad (5)$$

Equation (4) is still valid. This time, however, it is not merely a mathematical consequence of the definition of the emf ; rather, it is a true physical law deduced from experiment! Let us examine it in some detail.

In a region of space where a time-varying e/m field $(\vec{E}, \vec{B})$ exists, consider an arbitrary open surface *S* bounded by the closed curve *C*:



(The *relative* direction of $\overrightarrow{dl}$ and the surface element $\overrightarrow{da}$, normal to *S*, is determined according to the familiar right-hand rule.) The loop *C* is assumed stationary relative to the inertial observer; hence the emf along *C* at time *t* is given by (5). The magnetic flux through *S* at this instant is

$$\Phi_m(t) = \int_S \vec{B} \cdot \overrightarrow{da}$$

(Note that the signs of $\mathcal{E}$ and Φ_m depend on the chosen positive direction of *C*.) Since the field $\vec{B}$ is *solenoidal*, the value of Φ_m for a given *C* is independent of the choice of the surface *S*. That is, the same magnetic flux will go through *any* open surface bounded by the closed curve *C*.

According to the *Faraday-Henry law*,

$$\mathcal{E} = -\frac{d\Phi_m}{dt} \tag{6}$$

or explicitly,

$$\oint_C \vec{E} \cdot \overrightarrow{dl} = -\frac{d}{dt} \int_S \vec{B} \cdot \overrightarrow{da} \tag{7}$$

(The negative sign on the right-hand sides of (6) and (7) expresses *Lenz's law*.)

Equation (7) can be re-expressed in differential form by using Stokes' theorem,

$$\oint_C \vec{E} \cdot \overrightarrow{dl} = \int_S (\vec{\nabla} \times \vec{E}) \cdot \overrightarrow{da}$$

and by taking into account that the surface *S* may be arbitrarily chosen. The result is

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{8}$$

We note that if $\partial \vec{B} / \partial t \neq 0$, then necessarily $\vec{E} \neq 0$. Hence, as already mentioned, a time-varying magnetic field is always accompanied by an electric field. If, however, $\vec{B}$ is *static* ($\partial \vec{B} / \partial t = 0$), then $\vec{E}$ is *irrotational*: $\vec{\nabla} \times \vec{E} = 0 \Leftrightarrow \oint \vec{E} \cdot \overrightarrow{dl} = 0$, which allows for the possibility that $\vec{E} = 0$.

Corollary: The emf around a *fixed* loop *C* inside a *static* e/m field $\left(\vec{E}(\vec{r}),\, \vec{B}(\vec{r})\right)$ is $\mathcal{E} = 0$ (the student should explain this).

5. EMF OF A CIRCUIT CONTAINING A BATTERY AND A RESISTOR

Consider a circuit consisting of an ideal battery (i.e., one with no internal resistance) connected to an external resistor. As shown below, the emf of the circuit *in the direction of the current* is equal to the voltage *V* of the battery. Moreover, the emf in this case represents the *work per unit charge* done by the source (battery).

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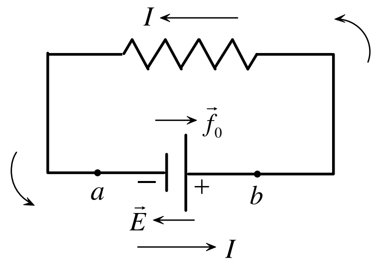


We recall that, in general, the emf of a circuit *C* at time *t* is equal to the integral

$$\mathcal{E} = \oint_C \vec{f} \cdot \overrightarrow{dl}$$

where $\vec{f} = \vec{F}/q$ is the force per unit charge at the location of the element $\overrightarrow{dl}$ of the circuit, at time *t*. In essence, we assume that in every element $\overrightarrow{dl}$ we have placed a test charge *q* (this could be, e.g., a free electron of the conducting part of the circuit). The force $\vec{F}$ on each *q* is then measured *simultaneously* for all charges at time *t*. Since here we are dealing with a *static* (time-independent) situation, however, we can treat the problem somewhat differently: The measurements of the forces $\vec{F}$ on the charges *q* need not be made at the same instant, given that nothing changes with time, anyway. So, instead of placing several charges *q* around the circuit and measuring the forces $\vec{F}$ on each of them at a particular instant, we imagine *a single charge q* making a complete tour around the loop *C*. We may assume, e.g., that the charge *q* is one of the (*conventionally positive*) free electrons taking part in the constant current *I* flowing in the circuit. We then measure the force $\vec{F}$ on *q* at each point of *C*.

We thus assume that *q* is a *positive* charge moving *in the direction of the current I*. We also assume that the direction of circulation of *C* is the *same as the direction of the current* (counterclockwise in the figure). During its motion, *q* is subject to two forces: (1) the force $\vec{F}_0$ by the source (battery) that carries *q* from the negative pole *a* to the positive pole *b through the source*, and (2) the electrostatic force $\vec{F}_e = q\vec{E}$ due to the electrostatic field $\vec{E}$ at each point of the circuit *C* (both inside and outside the source). The total force on *q* is

$$\vec{F} = \vec{F}_0 + \vec{F}_e = \vec{F}_0 + q\vec{E} \;\Rightarrow\; \vec{f} = \frac{\vec{F}}{q} = \frac{\vec{F}_0}{q} + \vec{E} \equiv \vec{f}_0 + \vec{E}$$

Then,

$$\mathcal{E} = \oint_C \vec{f} \cdot \overrightarrow{dl} = \oint_C \vec{f}_0 \cdot \overrightarrow{dl} + \oint_C \vec{E} \cdot \overrightarrow{dl} = \oint_C \vec{f}_0 \cdot \overrightarrow{dl} \tag{9}$$

since $\oint_C \vec{E} \cdot \overrightarrow{dl} = 0$ for an electrostatic field. However, the action of the source on *q* is limited to the region between the poles of the battery, that is, the section of the circuit from *a* to *b*. Hence, $\vec{f}_0 = 0$ *outside* the source, so that (9) reduces to

$$\mathcal{E} = \int_a^b \vec{f}_0 \cdot \overrightarrow{dl} \tag{10}$$

Now, since the current *I* is constant, the charge *q* moves at constant speed along the circuit. This means that the *total* force on *q* in the direction of the path *C* is zero. In the interior of the resistor, the electrostatic force $\vec{F}_e = q\vec{E}$ is counterbalanced by the force on *q* due to the collisions of the charge with the positive ions of the metal (this latter force does *not* contribute to the emf and is *not* counted in its evaluation!). In the interior of the (ideal) battery, however, where there is no resistance, the electrostatic force $\vec{F}_e$ must be counterbalanced by the *opposing* force $\vec{F}_0$ exerted by the source. Thus, in the section of the circuit between *a* and *b*,

$$\vec{F} = \vec{F}_0 + \vec{F}_e = 0 \;\Rightarrow\; \vec{f} = \frac{\vec{F}}{q} = \vec{f}_0 + \vec{E} = 0 \;\Rightarrow\; \vec{f}_0 = -\vec{E}$$

Equation (10) then takes the final form,

$$\mathcal{E} = -\int_a^b \vec{E}\cdot\overrightarrow{dl} = V_b - V_a = V \tag{11}$$

where V_a and V_b are the electrostatic potentials at *a* and *b*, respectively. This is, of course, what every student knows from elementary e/m courses!

The work done by the source on *q* upon transferring the charge from *a* to *b* is

$$W = \int_a^b \vec{F}_0\cdot\overrightarrow{dl} = q\int_a^b \vec{f}_0\cdot\overrightarrow{dl} = q\,\mathcal{E} \tag{12}$$

[where we have used (10)]. So, the *work of the source per unit charge* is $W/q=\mathcal{E}$. This work is converted into heat in the resistor, so that the source must again supply energy in order to carry the charges once more from *a* to *b*. This is something like the torture of Sisyphus in Greek mythology!

6. EMF AND OHM'S LAW

Consider a closed wire *C* inside an e/m field. The circuit may contain sources (e.g., a battery) and may also be in motion relative to our inertial frame of reference. Let *q* be a test charge at the location of the element $\overrightarrow{dl}$ of *C*, and let $\vec{F}$ be the total force on *q* (due to the e/m field and/or the sources) at time *t*. (As mentioned in Sec.2, this force is, classically, a frame-independent quantity.) The force per unit charge at the location of $\overrightarrow{dl}$ at time *t*, then, is $\vec{f} = \vec{F}/q$. According to our general definition, the emf of the circuit at time *t* is

$$\mathcal{E} = \oint_C \vec{f}\cdot\overrightarrow{dl} \tag{13}$$

Now, if σ is the *conductivity* of the wire, then, by *Ohm's law* in its general form (see, e.g., p. 285 of [1]) we have:

$$\vec{J} = \sigma\vec{f} \tag{14}$$

where $\vec{J}$ is the *volume current density* at the location of $\overrightarrow{dl}$ at time *t*. (Note that the more common expression $\vec{J}=\sigma\vec{E}$, found in most textbooks, is a special case of the above formula. Note also that $\vec{J}$ is measured *relative to the wire*, thus is the same for all inertial observers.) By combining (13) and (14) we get:

$$\mathcal{E}=\frac{1}{\sigma}\oint_C \vec{J}\cdot\overrightarrow{dl} \tag{15}$$

Taking into account that $\vec{J}$ is in the direction of $\overrightarrow{dl}$ at each point of *C*, we write:

$$\vec{J}\cdot\overrightarrow{dl}=J\,dl=\frac{I}{S}dl$$

where *S* is the *constant* cross-sectional area of the wire. If we make the additional assumption that, at each instant *t*, the current *I* is constant around the circuit (although *I* may vary with time), we finally get:

$$\mathcal{E}=\frac{l}{\sigma S}I=\frac{\rho l}{S}I=IR \tag{16}$$

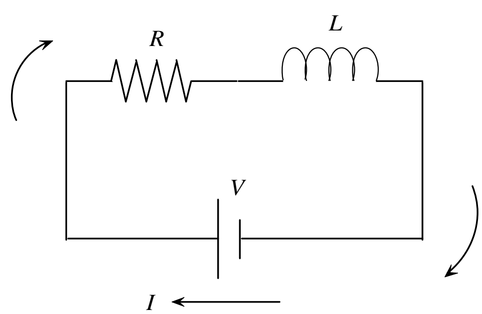
where *l* is the total length of the wire, $\rho=1/\sigma$ is the *resistivity* of the material, and *R* is the total resistance of the circuit. Equation (16) is the familiar special form of Ohm's law.

As an example, let us return to the circuit of Sec.5, this time assuming a *non-ideal* battery with internal resistance *r*. Let R_0 be the external resistance connected to the battery. The total resistance of the circuit is $R=R_0+r$. As before, we call $V=V_b-V_a$ the potential difference between the terminals of the battery, which is equal to the voltage across the external resistor. Hence, $V=IR_0$, where *I* is the current in the circuit. The emf of the circuit (in the direction of the current) is

$$\mathcal{E}=IR=I(R_0+r)=V+Ir$$

Note that the potential difference *V* between the terminals *a* and *b* equals the emf only when no current is flowing ($I=0$).

As another example, consider a circuit *C* containing an ideal battery of voltage *V* and having total resistance *R* and total *inductance L*:



In this case, the emf of *C in the direction of the current flow* is

$$\mathcal{E}(t) = V + V_L = V - L\frac{dI}{dt} = I(t)R$$

To understand why the total emf of the circuit is $V+V_L$, we think as follows: On its tour around the circuit, a test charge q is subject to two forces (ignoring collisions with the positive ions in the interior of the wire): a force *inside* the source, and a force by the *non-conservative* electric field accompanying the time-varying magnetic flux through the circuit. Hence, the total emf will be the sum of the emf due to the (ideal) battery alone and the emf expressed by the Faraday-Henry law (6). The latter emf is precisely V_L; it has a nonzero value for as long as the current I is changing.

Some interesting energy considerations are here in order. The total power supplied to the circuit by the battery at time t is

$$P = IV = I^2R + LI\frac{dI}{dt}$$

The term I^2R represents the power *irreversibly lost* as heat in the resistor (energy, per unit time, spent in moving the electrons through the crystal lattice of the conductor and transferred to the ions that make up the lattice). Thus, this power must necessarily be supplied back by the source in order to *maintain* the current against dissipative losses in the resistor. On the other hand, the term $LI\,(dI/dt)$ represents the energy per unit time required to *build up* the current against the "back emf" V_L. This energy is *retrievable* and is given back to the source when the current decreases. It may also be interpreted as energy per unit time required in order to *establish the magnetic field* associated with the current. This energy is "stored" in the magnetic field surrounding the circuit.

7. CONCLUDING REMARKS

In concluding this article, let us highlight a few points of importance:

1. The emf was defined as a line integral of force per unit charge around a loop (or "circuit") in an e/m field. The loop may or may not consist of a real conducting wire, and it may contain sources such as batteries.
2. In the classical (non-relativistic) limit, the emf is independent of the inertial frame of reference with respect to which it is measured.
3. In the case of *purely motional* emf, Faraday's "law" (4) is in essence a mere consequence of the definition of the emf. On the contrary, when a time-dependent magnetic field is present, the similar-looking equation (6) is a true physical law (the Faraday-Henry law).
4. In a DC circuit with a battery, the emf in the direction of the current equals the voltage of the battery and represents work per unit charge done by the source.
5. If the loop describing the circuit represents a conducting wire of finite resistance, Ohm's law can be expressed in terms of the emf by equation (16).

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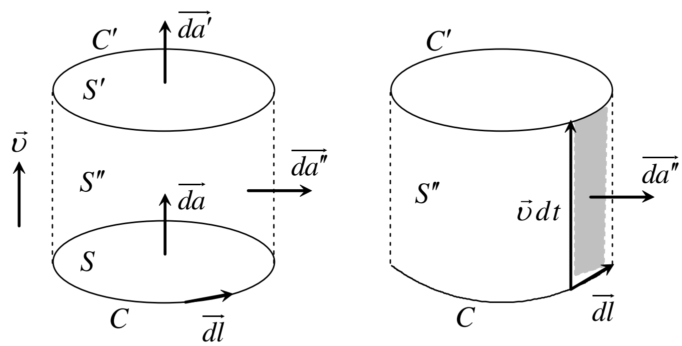
APPENDIX

Here is an analytical proof of equation (4) of Sec.3:

Assume that, at time *t*, the wire describes a closed curve *C* that is the boundary of a plane surface *S*. At time *t´= t+dt*, the wire (which has moved in the meanwhile) describes another curve *C´* that encloses a surface *S´*. Let $\vec{dl}$ be an element of *C* in the direction of circulation of the curve, and let $\vec{\upsilon}$ be the velocity of this element relative to an inertial observer (the velocity of the elements of *C* may vary along the curve):



The direction of the surface elements $\vec{da}$ and $\vec{da'}$ is consistent with the chosen direction of $\vec{dl}$, according to the right-hand rule. The element of the side ("cylindrical") surface *S´´* formed by the motion of *C*, is equal to

$$\vec{da''} = \vec{dl} \times (\vec{\upsilon}\; dt) = (\vec{dl} \times \vec{\upsilon})\, dt$$

Since the magnetic field is static, we can view the situation in a somewhat different way: Rather than assuming that the curve *C* moves within the time interval *dt* so that its points coincide with the points of the curve *C´* at time *t´*, we consider two *constant* curves *C* and *C´ at the same instant t*. In the case of a *static* field $\vec{B}$, the magnetic flux through *C´* at time *t´= t+dt* (according to our original assumption of a moving curve) is the same as the flux through this same curve at time *t*, given that no change of the magnetic field occurs within the time interval *dt*. Now, we note that the open surfaces $S_1=S$ and $S_2= S' \cup S''$ share a common boundary, namely, the curve *C*. Since the magnetic field is *solenoidal*, the same magnetic flux Φ_m passes through S_1 and S_2 at time *t*. That is,

$$\int_{S_1} \vec{B}\cdot\vec{da_1} = \int_{S_2} \vec{B}\cdot\vec{da_2} \;\;\Rightarrow\;\; \int_{S} \vec{B}\cdot\vec{da} = \int_{S'} \vec{B}\cdot\vec{da'} + \int_{S''} \vec{B}\cdot\vec{da''}$$

But, returning to our initial assumption of a *moving* curve, we note that

$$\int_S \vec{B}\cdot\vec{da} = \Phi_m(t) = \textit{magnetic flux through the wire at time t}$$

and

$$\int_{S'} \vec{B} \cdot \overrightarrow{da'} = \Phi_m(t+dt) = \textit{ magnetic flux through the wire at time t+dt}$$

Hence,

$$\Phi_m(t) = \Phi_m(t+dt) + \int_{S''} \vec{B} \cdot \overrightarrow{da''} \;\; \Rightarrow$$

$$d\Phi_m = \Phi_m(t+dt) - \Phi_m(t) = -\int_{S''} \vec{B} \cdot \overrightarrow{da''} = -\,dt \oint_C \vec{B} \cdot (\overrightarrow{dl} \times \vec{\upsilon}) \;\; \Rightarrow$$

$$-\frac{d\Phi_m}{dt} = \oint_C \vec{B} \cdot (\overrightarrow{dl} \times \vec{\upsilon}) = \oint_C (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} = \mathcal{E}$$

in accordance with (3) and (4).

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[1] One of us (C.J.P.) strongly feels that the 2nd Edition of 1975 (unfortunately out of print) was a much better edition!

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Does the electromotive force (always) represent work?

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Abstract

In the literature of Electromagnetism, the electromotive force of a "circuit" is often defined as work done on a unit charge during a complete tour of the latter around the circuit. We explain why this statement cannot be generally regarded as true, although it is indeed true in certain simple cases. Several examples are used to illustrate these points.

1. Introduction

In a recent paper [1] the authors suggested a pedagogical approach to the *electromotive force* (emf) of a "circuit", a fundamental concept of Electromagnetism. Rather than defining the emf in an *ad hoc* manner for each particular electrodynamic system, this approach begins with the most general definition of the emf and then specializes to certain cases of physical interest, thus recovering the familiar expressions for the emf.

Among the various examples treated in [1], the case of a simple battery-resistor circuit was of particular interest since, in this case, the emf was shown to be equal to the *work, per unit charge*, done by the source (battery) for a complete tour around the circuit. Now, in the literature of Electrodynamics the emf is often *defined* as work per unit charge. As we explain in this paper, this is not generally true except for special cases, such as the aforementioned one.

In Section 2, we give the general definition of the emf, $\mathcal{E}$, and, separately, that of the work per unit charge, w, done by the agencies responsible for the generation and preservation of a current flow in the circuit. We then state the necessary conditions in order for the equality $\mathcal{E}=w$ to hold. We stress that, by their very definitions, $\mathcal{E}$ and w are *different* concepts. Thus, the equation $\mathcal{E}=w$ suggests the possible equality of the *values* of two physical quantities, not the conceptual identification of these quantities!

Section 3 reviews the case of a circuit consisting of a battery connected to a resistive wire, in which case the equality $\mathcal{E}=w$ is indeed valid.

In Sec. 4, we study the problem of a wire moving through a static magnetic field. A particular situation where the equality $\mathcal{E}=w$ is valid is treated in Sec. 5.

Finally, Sec. 6 examines the case of a stationary wire inside a time-varying magnetic field. It is shown that the equality $\mathcal{E}=w$ is satisfied only in the special case where the magnetic field varies linearly with time.

2. The general definitions of emf and work per unit charge

Consider a region of space in which an electromagnetic (e/m) field exists. In the most general sense, any *closed* path C (or *loop*) within this region will be called a *"circuit"* (whether or not the whole or parts of C consist of material objects such as wires, resistors, capacitors, batteries, etc.). We *arbitrarily* assign a positive direction of traversing the loop C, and we consider an element $\vec{dl}$ of C oriented in the positive direction (Fig. 1).

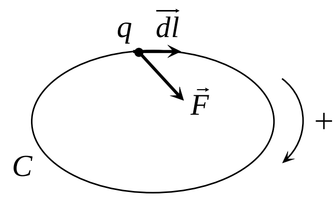

Figure 1: An oriented loop representing a circuit.

Imagine now a test charge q located at the position of $\vec{dl}$, and let $\vec{F}$ be the force on q at time t. This force is exerted by the e/m field itself, as well as, possibly, by additional *energy sources* (e.g., batteries or some external mechanical action) that may contribute to the generation and preservation of a current flow around the loop C. The *force per unit charge* at the position of $\vec{dl}$ at time t, is

$$\vec{f} = \frac{\vec{F}}{q} \tag{1}$$

Note that $\vec{f}$ is independent of q, since the electromagnetic force on q is proportional to the charge. In particular, reversing the sign of q will have no effect on $\vec{f}$ (although it will change the direction of $\vec{F}$).

In general, neither the shape nor the size of C is required to remain fixed. Moreover, the loop may be in motion relative to an external inertial observer. Thus, for a loop of (possibly) variable shape, size or position in space, we will use the notation $C(t)$ to indicate the state of the curve at time t.

We now define the *electromotive force* (emf) of the circuit C *at time t* as the line integral of $\vec{f}$ along C, taken in the *positive* sense of C:

$$\mathcal{E}(t) = \oint_{C(t)} \vec{f}(\vec{r},t)\cdot \overrightarrow{dl} \tag{2}$$

(where $\vec{r}$ is the position vector of $\overrightarrow{dl}$ relative to the origin of our coordinate system). Note that the sign of the emf is dependent upon our choice of the positive direction of circulation of C: by changing this convention, the sign of $\mathcal{E}$ is reversed.

As mentioned above, the force (per unit charge) defined in (1) can be attributed to two factors: the interaction of q with the e/m field itself and the action on q due to any additional energy sources. Eventually, this latter interaction is *electromagnetic* in nature even when it originates from some external mechanical action. We write:

$$\vec{f} = \vec{f}_{em} + \vec{f}_{app} \tag{3}$$

where $\vec{f}_{em}$ is the force due to the e/m field and $\vec{f}_{app}$ is the *applied force* due to an additional energy source. We note that the force (3) does not include any *resistive* (dissipative) forces that oppose a charge flow along C; it only contains forces that may contribute to the generation and preservation of such a flow in the circuit.

Now, suppose we allow *a single charge q* to make a full trip around the circuit C under the action of the force (3). In doing so, the charge describes a curve C' *in space* (not necessarily a closed one!) relative to an external inertial observer. Let $\overrightarrow{dl'}$ be an element of C' representing an infinitesimal displacement of q in space, in time dt. We define the *work per unit charge* for this complete tour around the circuit by the integral:

$$w = \int_{C'} \vec{f}\cdot\overrightarrow{dl'} \tag{4}$$

For a *stationary* circuit of *fixed* shape, C' coincides with the closed curve C and (4) reduces to

$$w = \oint_C \vec{f}\cdot\overrightarrow{dl} \qquad (\textit{fixed } C) \tag{5}$$

It should be noted carefully that the integral (2) is evaluated *at a fixed time t*, while in the integrals (4) and (5) time is allowed to flow! In general, the value of w depends on the time t_0 and the point P_0 at which q starts its round trip on C. Thus, there is a certain ambiguity in the definition of work per unit charge. On the other hand, the ambiguity (so to speak) with respect to the emf is related to the dependence of the latter on time t.

The question now is: can the emf be equal *in value* to the work per unit charge, despite the fact that these quantities are defined differently? For the equality $\mathcal{E}=w$ to hold, both $\mathcal{E}$ and w must be defined unambiguously. Thus, $\mathcal{E}$ must be *constant*, independent of time ($d\mathcal{E}/dt=0$) while w must not depend on the initial time t_0 or the initial point P_0 of the round trip of q on C. These requirements are *necessary conditions* in order for the equality $\mathcal{E}=w$ to be meaningful.

In the following sections we illustrate these ideas by means of several examples. As will be seen, the satisfaction of the above-mentioned conditions is the exception rather than the rule!

3. A resistive wire connected to a battery

Consider a circuit consisting of an ideal battery (i.e., one with no internal resistance) connected to a metal wire of total resistance R (Fig. 2). As shown in [1] (see also [2]), the emf of the circuit *in the direction of the current* is equal to the voltage V of the battery. Moreover, the emf in this case represents the work, per unit charge, done by the source (battery). Let us review the proof of these statements.

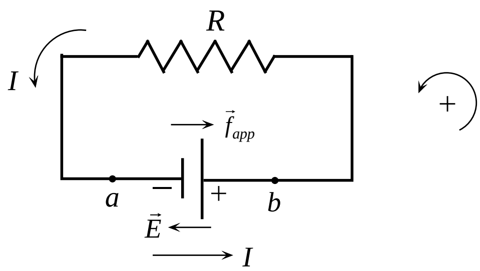

Figure 2: A battery connected to a resistive wire.

A (*conventionally positive*) moving charge q is subject to two forces around the circuit C: an electrostatic force $\vec{F}_e = q\vec{E}$ at every point of C and a force $\vec{F}_{app}$ *inside* the battery, the latter force carrying q from the negative pole a to the positive pole b *through the source*. According to (3), the total force per unit charge is

$$\vec{f} = \vec{f}_e + \vec{f}_{app} = \vec{E} + \vec{f}_{app} \ .$$

The emf in the direction of the current (i.e., counterclockwise), at any time t, is

$$\begin{aligned}\mathcal{E} &= \oint_C \vec{f}\cdot\overrightarrow{dl} \\ &= \oint_C \vec{E}\cdot\overrightarrow{dl} + \oint_C \vec{f}_{app}\cdot\overrightarrow{dl} \\ &= \int_a^b \vec{f}_{app}\cdot\overrightarrow{dl}\end{aligned} \tag{6}$$

where we have used the facts that $\oint_C \vec{E}\cdot\overrightarrow{dl}=0$ for an electrostatic field and that the action of the source on q is limited to the region between the poles of the battery.

Now, in a steady-state situation (I = constant) the charge q moves at constant speed along the circuit. This means that the total force on q in the direction of the path C is zero. In the interior of the wire, the electrostatic force $\vec{F}_e = q\vec{E}$ is counterbalanced by the resistive force on q due to the collisions of the charge with the positive ions of the metal (as mentioned previously, this latter force does *not* contribute to the emf). In the interior of the (ideal) battery, however, where there is no resistance, the electrostatic force must be counterbalanced by the opposing force exerted by the source. Thus, in the section of the circuit between a and b, $\vec{f}_{app} = -\vec{f}_e = -\vec{E}$. By (6), then, we have:

$$\mathcal{E} = -\int_a^b \vec{E}\cdot\overrightarrow{dl} = V_b - V_a = V \tag{7}$$

where V_a and V_b are the electrostatic potentials at a and b, respectively. We note that the emf is constant in time, as expected in a steady-state situation.

Next, we want to find the work per unit charge for a complete tour around the circuit. To this end, we allow *a single charge q* to make a full trip around C and we use expression (5) (since the wire is stationary and of fixed shape). In applying this relation, time is assumed to flow as q moves along C. Given that the situation is static (time-independent), however, time is not really an issue since it doesn't matter at what moment the charge will pass by any given point of C. Thus, the integration in (5) will yield the same result (7) as the integration in (6), despite the fact that, in the latter case, time was assumed *fixed*. We conclude that the equality $w=\mathcal{E}$ is valid in this case: the emf *does* represent work per unit charge.

4. Moving wire inside a static magnetic field

Consider a wire C moving in the xy-plane. The shape and/or size of the wire need not remain fixed during its motion. A static magnetic field $\vec{B}(\vec{r})$ is present in the region of space where the wire is moving. For simplicity, we assume that this field is normal to the plane of the wire and directed *into* the page.

In Fig. 3, the z-axis is normal to the plane of the wire and directed towards the reader. We call $\overrightarrow{da}$ an infinitesimal normal vector representing an element of the plane surface bounded by the wire (this vector is directed *into* the plane, consistently with the chosen clockwise direction of traversing the loop C). If $\hat{u}_z$ is the unit vector on the z-axis, then $\overrightarrow{da} = -(da)\,\hat{u}_z$ and $\vec{B} = -B(\vec{r})\,\hat{u}_z$, where $B(\vec{r}) = |\,\vec{B}(\vec{r})\,|$.

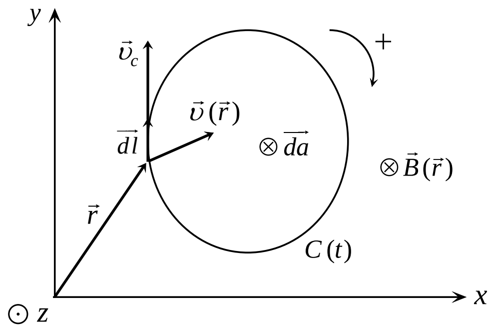

Figure 3: A wire C moving inside a static magnetic field.

Consider an element $\overrightarrow{dl}$ of the wire, located at a point with position vector $\vec{r}$ relative to the origin of our inertial frame of reference. Call $\vec{\upsilon}(\vec{r})$ the velocity of this element relative to our frame. Let q be a (*conventionally positive*) charge passing by the considered point at time t. This charge executes a composite motion, having a velocity $\vec{\upsilon}_c$ *along the wire* and acquiring an extra velocity $\vec{\upsilon}(\vec{r})$ due to the motion of the wire itself. The total velocity of q relative to us is $\vec{\upsilon}_{tot} = \vec{\upsilon}_c + \vec{\upsilon}$.

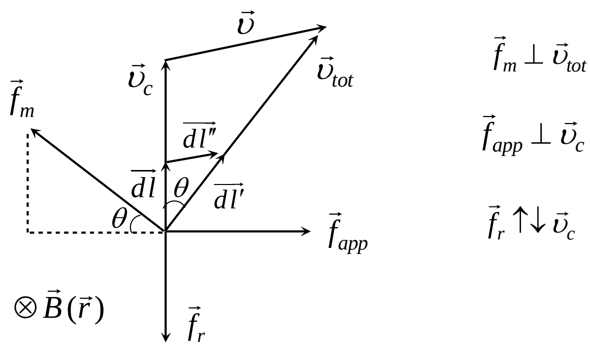

Figure 4: Balance of forces per unit charge.

The balance of forces acting on q is shown in the diagram of Fig. 4. The *magnetic force* on q is normal to the charge's total velocity and equal to $\vec{F}_m = q\,(\vec{\upsilon}_{tot}\times\vec{B})$. Hence, the magnetic force per unit charge is $\vec{f}_m = \vec{\upsilon}_{tot}\times\vec{B}$. Its component along the wire (i.e., in the direction of $\overrightarrow{dl}$) is counterbalanced by the *resistive force* $\vec{f}_r$, which opposes the motion of q along C (this force, as mentioned previously, does *not* contribute to the emf). However, the component of the magnetic force *normal* to the wire will tend to make the wire move "backwards" (in a direction opposing the desired motion of the wire) unless it is counterbalanced by some *external* mechanical action (e.g., our hand, which pulls the wire forward). Now, the charge q takes a share of this action by means of some force transferred to it by the structure of the wire. This force (which will be called an *applied force*) must be *normal* to the wire (in order to counterbalance the normal component of the magnetic force). We denote the applied force per unit charge by $\vec{f}_{app}$. Although this force originates from an external mechanical action, it is delivered to q through an *electromagnetic* interaction with the crystal lattice of the wire (not to be confused with the resistive force, whose role is different!).

According to (3), the total force contributing to the emf of the circuit is $\vec{f} = \vec{f}_m + \vec{f}_{app}$. By (2), the emf at time t is

$$\mathcal{E}(t) = \oint_{C(t)} \vec{f}_m \cdot \overrightarrow{dl} + \oint_{C(t)} \vec{f}_{app} \cdot \overrightarrow{dl} \ .$$

The second integral vanishes since the applied force is normal to the wire element at every point of C. The integral of the magnetic force is equal to

$$\oint_C (\vec{\upsilon}_{tot} \times \vec{B}) \cdot \overrightarrow{dl} = \oint_C (\vec{\upsilon}_c \times \vec{B}) \cdot \overrightarrow{dl} + \oint_C (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} \ .$$

The first integral on the right vanishes, as can be seen by inspecting Fig. 4. Thus, we finally have:

$$\mathcal{E}(t) = \oint_{C(t)} [\vec{\upsilon}(\vec{r}) \times \vec{B}(\vec{r})] \cdot \overrightarrow{dl} \tag{8}$$

As shown analytically in [1, 2], the emf of C is equal to

$$\mathcal{E}(t) = -\frac{d}{dt} \Phi_m(t) \tag{9}$$

where we have introduced the *magnetic flux* through C,

$$\Phi_m(t) = \int_{S(t)} \vec{B}(\vec{r}) \cdot \overrightarrow{da} = \int_{S(t)} B(\vec{r})\, da \tag{10}$$

[By $S(t)$ we denote *any* open surface bounded by C at time t; e.g., the plane surface enclosed by the wire.]

Now, let C' be the path of q in space relative to the external observer, for a full trip of q around the wire (in general, C' will be an *open* curve). According to (4), the work done per unit charge for this trip is

$$w = \int_{C'} \vec{f}_m \cdot \overrightarrow{dl'} + \int_{C'} \vec{f}_{app} \cdot \overrightarrow{dl'} \ .$$

The first integral vanishes (cf. Fig. 4), while for the second one we notice that

$$\vec{f}_{app} \cdot \overrightarrow{dl'} = \vec{f}_{app} \cdot \overrightarrow{dl} + \vec{f}_{app} \cdot \overrightarrow{dl''} = \vec{f}_{app} \cdot \overrightarrow{dl''}$$

(since the applied force is normal to the wire element everywhere; see Fig. 4). Thus we finally have:

$$w = \int_{C'} \vec{f}_{app} \cdot \overrightarrow{dl'} \tag{11a}$$

with

$$\vec{f}_{app} \cdot \overrightarrow{dl'} = \vec{f}_{app} \cdot \overrightarrow{dl''} = \vec{f}_{app} \cdot \vec{\upsilon}\, dt \tag{11b}$$

where $\overrightarrow{dl''} = \vec{\upsilon}\, dt$ is the infinitesimal displacement of the wire element in time dt.

5. An example: Motion inside a uniform magnetic field

Consider a metal bar (ab) of length h, sliding parallel to itself with constant speed υ on two parallel rails that form part of a U-shaped wire, as shown in Fig. 5. A *uniform* magnetic field $\vec{B}$, pointing into the page, fills the entire region.

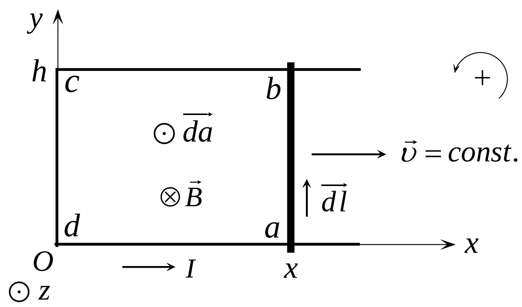

Figure 5: A metal bar (ab) sliding on two parallel rails that form part of a U-shaped wire.

A circuit $C(t)$ of variable size is formed by the rectangular loop ($abcda$). The field and the surface element are written, respectively, as $\vec{B} = -B\,\hat{u}_z$ (where $B = |\vec{B}| = const.$) and $\overrightarrow{da} = (da)\,\hat{u}_z$ (note that the direction of traversing the loop C is now counterclockwise).

The general diagram of Fig. 4, representing the balance of forces, reduces to the one shown in Fig. 6. Note that this latter diagram concerns only the *moving* part (ab) of the circuit, since it is in this part only that the velocity $\vec{\upsilon}$ and the applied force $\vec{f}_{app}$ are nonzero.

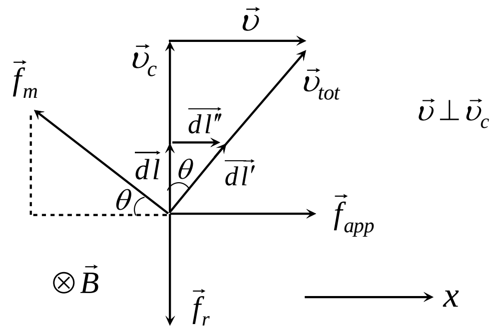

Figure 6: Balance of forces per unit charge.

The emf of the circuit at time t is, according to (8),

$$\mathcal{E}(t) = \oint_{C(t)} (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl}$$

$$= \int_a^b \upsilon B\, dl = \upsilon B \int_a^b dl = \upsilon B h \ .$$

Alternatively, the magnetic flux through *C* is

$$\Phi_m(t) = \int_{S(t)} \vec{B}(\vec{r}) \cdot \overrightarrow{da} = -\int_{S(t)} B\, da = -B \int_{S(t)} da$$
$$= -Bhx$$

(where *x* is the momentary position of the bar at time *t*), so that

$$\mathcal{E}(t) = -\frac{d}{dt}\Phi_m(t) = Bh\frac{dx}{dt} = Bh\upsilon \ .$$

We note that the emf is constant (time-independent).

Next, we want to use (11) to evaluate the work per unit charge for a complete tour of a charge around *C*. Since the applied force is nonzero only on the section (*ab*) of *C*, the path of integration, C' (which is a straight line, given that the charge moves at constant velocity in space) will correspond to the motion of the charge along the metal bar only, i.e., from *a* to *b*. (Since the bar is being displaced in space while the charge is traveling along it, the line C' will *not* be parallel to the bar.) According to (11),

$$w = \int_{C'} \vec{f}_{app} \cdot \overrightarrow{dl'} \quad \text{with}$$
$$\vec{f}_{app} \cdot \overrightarrow{dl'} = \vec{f}_{app} \cdot \overrightarrow{dl''} = f_{app}\, dl'' = f_{app}\, \upsilon\, dt$$

(cf. Fig. 6). Now, the role of the applied force is to counterbalance the *x*-component of the magnetic force in order that the bar may move at constant speed in the *x* direction. Thus,

$$f_{app} = f_m \cos\theta = \upsilon_{tot} B \cos\theta = B\upsilon_c$$

and

$$f_{app}\, \upsilon\, dt = B\upsilon\upsilon_c\, dt = B\upsilon\, dl$$

(since $\upsilon_c\, dt$ represents an elementary displacement *dl* of the charge along the metal bar in time *dt*). We finally have:

$$w = \int_a^b B\upsilon\, dl = B\upsilon \int_a^b dl = B\upsilon h \ .$$

We note that, in this specific example, the value of the work per unit charge is equal to that of the emf, both these quantities being constant and unambiguously defined. This would *not* have been the case, however, if the magnetic field were *nonuniform*!

6. Stationary wire inside a time-varying magnetic field

Our final example concerns a *stationary* wire *C* inside a *time-varying* magnetic field of the form $\vec{B}(\vec{r},t) = -B(\vec{r},t)\,\hat{u}_z$ (where $B(\vec{r},t) = |\vec{B}(\vec{r},t)|$), as shown in Fig. 7.

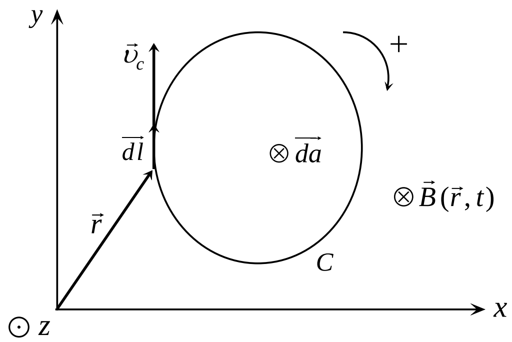

Figure 7: A stationary wire *C* inside a time-varying magnetic field.

As is well known [1-7], the presence of a time-varying magnetic field implies the presence of an electric field $\vec{E}$ as well, such that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{12}$$

As discussed in [1], the emf of the circuit at time *t* is given by

$$\mathcal{E}(t) = \oint_C \vec{E}(\vec{r},t) \cdot \vec{dl} = -\frac{d}{dt}\Phi_m(t) \tag{13}$$

where

$$\Phi_m(t) = \int_S \vec{B}(\vec{r},t) \cdot \overrightarrow{da} = \int_S B(\vec{r},t)\, da \tag{14}$$

is the magnetic flux through *C* at this time.

On the other hand, the work per unit charge for a full trip around *C* is given by (5): $w = \oint_C \vec{f} \cdot \overrightarrow{dl}$, where $\vec{f} = \vec{f}_{em} = \vec{E} + (\vec{\upsilon}_c \times \vec{B})$, so that

$$w = \oint_C \vec{E} \cdot \overrightarrow{dl} + \oint_C (\vec{\upsilon}_c \times \vec{B}) \cdot \overrightarrow{dl} \ .$$

As is easy to see (cf. Fig. 7), the second integral vanishes, thus we are left with

$$w = \oint_C \vec{E} \cdot \overrightarrow{dl} \tag{15}$$

The similarity of the integrals in (13) and (15) is deceptive! The integral in (13) is evaluated *at a fixed time t*, while in (15) time is allowed to flow as the charge moves along *C*. Is it, nevertheless, possible that the *values* of these integrals coincide? As mentioned at the end of Sec. 2, a necessary condition for this to be the case is that the two integrations yield time-independent results. In order that $\mathcal{E}$ be time-independent (but nonzero), the magnetic flux (14) – thus the magnetic field itself – must increase *linearly* with time. On the other hand, the integration (15) for *w* will be time-independent if so is the electric field. By (12), then, the magnetic field must be linearly dependent on time, which brings us back to the previous condition.

As an example, assume that the magnetic field is of the form

$$\vec{B} = -B_0\, t\, \hat{u}_z \qquad (B_0 = const.)\,.$$

A possible solution of (12) for $\vec{E}$ is, in cylindrical coordinates,

$$\vec{E} = \frac{B_0 \rho}{2}\, \hat{u}_\varphi \;.$$

[We assume that these solutions are valid in a limited region of space (e.g., in the interior of a solenoid whose axis coincides with the *z*-axis) so that ρ is finite in the region of interest.] Now, consider a circular wire *C* of radius *R*, centered at the origin of the *xy*-plane. Then, given that $\vec{dl} = -(dl)\,\hat{u}_\varphi$,

$$\mathcal{E} = \oint_C \vec{E} \cdot \vec{dl} = -\frac{B_0 R}{2} \oint_C dl = -B_0 \pi R^2 .$$

Alternatively,

$$\Phi_m = \int_S B da = B_0 \pi R^2 t \,,$$

so that $\mathcal{E} = -\,d\,\Phi_m / dt = -B_0 \pi R^2$. We anticipate that, due to the time constancy of the electric field, the same result will be found for the work *w* by using (15).

7. Concluding remarks

No single, universally accepted definition of the emf seems to exist in the literature of Electromagnetism. The definition given in this article (as well as in [1]) comes close to those of [2] and [3]. In particular, by using an example similar to that of Sec. 5 in this paper, Griffiths [2] makes a clear distinction between the concepts of emf and work per unit charge. In [4] and [5] (as well as in numerous other textbooks) the emf is identified with work per unit charge, in general, while in [6] and [7] it is defined as a closed line integral of the non-conservative part of the electric field that accompanies a time-varying magnetic flux.

The balance of forces and the origin of work in a conducting circuit moving through a magnetic field are nicely discussed in [2, 8, 9]. An interesting approach to the relation between work and emf, utilizing the concept of virtual work, is described in [10].

Of course, the list of references cited above is by no means exhaustive. It only serves to illustrate the diversity of ideas concerning the concept of the emf. The subtleties inherent in this concept make it an interesting subject of study for both the researcher and the advanced student of classical Electrodynamics.

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Some aspects of the electromotive force: Educational review article

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Synopsis

Certain aspects of the concept of the electromotive force (emf) of a “circuit”, as this concept was defined in recent publications, are discussed. In particular, the independence of the emf from the conductivity of the circuit is explained and the role of the applied force in motional emf is analyzed.

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SOME ASPECTS OF THE

ELECTROMOTIVE FORCE

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Some aspects of the electromotive force

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Certain aspects of the concept of the electromotive force (emf) of a "circuit", as this concept was defined in recent publications, are discussed. In particular, the independence of the emf from the conductivity of the circuit is explained and the role of the applied force in motional emf is analyzed.

1. Definition and analytical expression of the emf

In recent articles [1,2] we studied the concept of the *electromotive force* (*emf*) of a "circuit" and examined the extent to which the emf represents work per unit charge for a complete tour around the circuit. This educational note contains some additional remarks regarding the emf; it may be regarded as an addendum to the aforementioned publications.

We consider a closed path C (or *loop*) in a region of space where an electromagnetic (e/m) field exists (Fig. 1). Generally speaking, this loop will be called a *"circuit"* if a charge flow can be sustained on it. We *arbitrarily* assign a positive direction of traversing the loop C and we consider an element $\vec{dl}$ of C oriented in the positive direction.

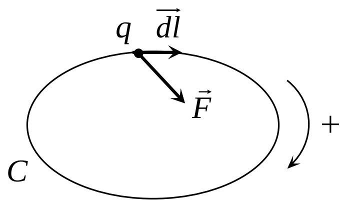

Figure 1

Let q be a *test charge*, which at time t is located at the position of $\vec{dl}$, and let $\vec{F}$ be the force on q at this time. The force $\vec{F}$ is exerted by the e/m field itself as well as, possibly, by additional *energy sources* (such as batteries or some external mechanical action) that may contribute to the generation and preservation of a current around the loop C. The *force per unit charge* at the position of $\vec{dl}$, at time t, is $\vec{f} = \vec{F}/q$. We note that $\vec{f}$ is independent of q since the e/m force on a charge is proportional to the charge.

Since, in general, neither the shape nor the size of C is required to remain fixed, and since the loop may also be in motion relative to an external observer, we will use the notation $C(t)$ to indicate the state, at time t, of a circuit of generally variable shape, size or position in space.

The *electromotive force* (*emf*) of the circuit C *at time t* is defined as the line integral of $\vec{f}$ along C, taken in the *positive* sense of C:

$$\mathcal{E}(t) = \oint_{C(t)} \vec{f}(\vec{r},t) \cdot \overrightarrow{dl} \tag{1}$$

where $\vec{r}$ is the position vector of $\overrightarrow{dl}$ relative to the origin of our coordinate system. Obviously, the sign of the emf is dependent upon our choice of the positive direction of circulation of C. It should be noted carefully that the integral (1) is evaluated *at a given time t*. Thus, the force $\vec{f}$ must be measured *simultaneously*, at time t, at all points of C.

The force $\vec{f}$ can be attributed to two factors: (*a*) the interaction of q with the existing e/m field itself; and (*b*) the action on q by any additional energy sources that may be necessary in order to maintain a steady flow of charge on C. (This latter interaction also is *electromagnetic* in nature, even when it originates from some external mechanical action.) We write

$$\vec{f} = \vec{f}_{em} + \vec{f}_{app} \tag{2}$$

where $\vec{f}_{em}$ is the force due to the e/m field and $\vec{f}_{app}$ is the *applied force* due to an additional energy source.

Two familiar cases of emf-driven circuits where an additional applied force is required are the following:

1. In a battery-resistor circuit [1-3] an applied force is necessary in order to carry a (conventionally *positive*) mobile charge from the negative to the positive pole of the battery, *through* the source. This force is provided by the battery itself.

2. In the case of a closed metal wire C moving in a time-independent magnetic field [2-5] the current on C is sustained for as long as the motion of C continues. This, in turn, necessitates the action of an external force on C (say, by our hand), as will be explained in Sec. 4.

Now, by (1) and (2),

$$\mathcal{E}(t) = \oint_{C(t)} \vec{f}_{em} \cdot \overrightarrow{dl} + \oint_{C(t)} \vec{f}_{app} \cdot \overrightarrow{dl} \equiv \mathcal{E}_{em}(t) + \mathcal{E}_{app}(t) \tag{3}$$

We would like to find an analytical expression for $\mathcal{E}_{em}(t)$. So, let $\left(\vec{E}(\vec{r},t), \vec{B}(\vec{r},t)\right)$ be the e/m field in the region of space where the loop $C(t)$ is lying. Let q be a test charge located, at time t, at the position of $\overrightarrow{dl}$ and let $\vec{\upsilon}_{tot}$ be the total velocity of q in space, relative to some inertial frame of reference. We write

$$\vec{\upsilon}_{tot} = \vec{\upsilon} + \vec{\upsilon}_c$$

where $\vec{\upsilon}_c$ is the velocity of q along C (i.e., in a direction parallel to $\overrightarrow{dl}$) while $\vec{\upsilon}$ is the velocity of $\overrightarrow{dl}$ itself due to a possible motion in space, or just a deformation over time, of the loop $C(t)$ as a whole. The total e/m force on q is

$$\vec{F}_{em} = q[\vec{E} + (\vec{\upsilon}_{tot} \times \vec{B})] \ ,$$

so that

$$\vec{f}_{em} = \frac{\vec{F}}{q} = \vec{E} + [(\vec{\upsilon} + \vec{\upsilon}_c) \times \vec{B}] \ .$$

Hence,

$$\mathcal{E}_{em}(t) = \oint_{C(t)} \vec{E} \cdot \overrightarrow{dl} + \oint_{C(t)} (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} + \oint_{C(t)} (\vec{\upsilon}_c \times \vec{B}) \cdot \overrightarrow{dl} \ .$$

Given that $\vec{\upsilon}_c$ is parallel to $\overrightarrow{dl}$, the last integral on the right vanishes. Thus, finally,

$$\mathcal{E}_{em}(t) = \oint_{C(t)} \vec{E}(\vec{r},t) \cdot \overrightarrow{dl} + \oint_{C(t)} [\vec{\upsilon}(\vec{r},t) \times \vec{B}(\vec{r},t)] \cdot \overrightarrow{dl} \equiv \mathcal{E}_e(t) + \mathcal{E}_m(t) \qquad (4)$$

We note that, in our definition of the emf, the force per unit charge was defined as $\vec{f} = \vec{F}/q$, assuming that a replica of a test charge q is placed at every point of the circuit and that the forces $\vec{F}$ on all test charges are measured *simultaneously* at time t. Now, in the case of a conducting loop C (say, a metal wire) it is reasonable to identify q with one of the (conventionally positive) mobile free electrons. This particular identification, although logical for practical purposes, is nevertheless not necessary, given that the force $\vec{f}$ is eventually independent of q. Thus, in general, q may just be considered as a *hypothetical* test charge that is not necessarily identified with an actual mobile charge.

2. Independence from conductivity

Let $C(t)$ be a conducting loop (say, a metal wire) inside a given e/m field. The emf of C at time t is given by (3) and (4). We note from (4) that the part $\mathcal{E}_{em}$ of the total emf is independent of the velocity $\vec{\upsilon}_c$ of q *along* C (where q may be conveniently – although not necessarily – assumed to be a mobile free electron of the conductor, conventionally considered as a *positive* charge). We may physically interpret this as follows:

The e/m field creates an emf $\mathcal{E}_{em}$ that tends to generate a charge flow on C. However, this emf does not by itself determine *how fast* the mobile charges move along C. Presumably, this will depend on physical properties of the path C that are associated with its *conductivity*. (For example, in a battery-resistance circuit the potential difference at the ends of the resistance – thus the value of the electric field inside the conductor – does not by itself determine the velocity $\vec{\upsilon}_c$ of the mobile charges along the circuit, since this velocity is related to the current generated by the source, which current depends, in turn, on the resistance of the circuit, according to Ohm's law.)

Now, the role of the part $\mathcal{E}_{app}$ of the total emf (3) is to *maintain* the charge flow on $C(t)$ that is generated by $\mathcal{E}_{em}$. We thus anticipate that $\mathcal{E}_{app}$ will also be independent of $\vec{\upsilon}_c$ (this is, e.g., the case in our previous example, where $\mathcal{E}_{app}$ is equal to the voltage of the battery [1-3]). In conclusion,

> *the total emf $\mathcal{E}(t)$ of a conducting loop $C(t)$ is not dependent upon the velocity of motion of the mobile charges q along the loop.*

This leads us to a further conclusion:

> *The total emf $\mathcal{E}(t)$ of a conducting loop $C(t)$ inside an e/m field is not dependent upon the conductivity of the loop.*

This can be justified by noting that, by its definition, the force (2) does not include contributions from *resistive forces* that oppose a charge flow on C; it only contains e/m interactions that may contribute to the generation and preservation of a current in the circuit. Note, however, that the *current* itself *does* depend on the *conductivity* σ of C, according to Ohm's law ($\vec{J} = \sigma \vec{f}$) [3].

Alternatively, as argued above, the emf does not depend on $\vec{\upsilon}_c$. Now, in a steady-state situation under given electrodynamic conditions (thus, for a given $\vec{f}$) this velocity is a linear function of the *mobility* μ of q, according to the empirical relation $\vec{\upsilon}_c = \mu \vec{f}$ (by which Ohm's law is deduced). On the other hand, the conductivity of C is given by $\sigma = qn\mu$. The *density* n of mobile charges, as well as the value of q, cannot affect the value of the emf since that quantity is defined per unit charge. We thus conclude that the emf of C cannot depend on μ, as well as on n and q; hence, $\mathcal{E}$ is independent of σ.

3. Emf and the Faraday-Henry law

Consider a region of space in which a (generally time-dependent) e/m field $(\vec{E}, \vec{B})$ exists. Let C be a *fixed* conducting loop in this region. There is no additional applied force on C, so (3) reduces to $\mathcal{E}(t) = \mathcal{E}_{em}(t)$. Furthermore, since C is stationary, $\vec{\upsilon}(\vec{r}, t)$ vanishes identically and, by (4), $\mathcal{E}_m(t) = 0$ and $\mathcal{E}_{em}(t) = \mathcal{E}_e(t)$. Thus, finally,

$$\mathcal{E}(t) = \oint_C \vec{E}(\vec{r}, t) \cdot \overrightarrow{dl} \qquad (5)$$

By Stokes' theorem,

$$\oint_C \vec{E} \cdot \overrightarrow{dl} = \int_S (\vec{\nabla} \times \vec{E}) \cdot \overrightarrow{da}$$

where S is any open surface bounded by C (Fig. 2).

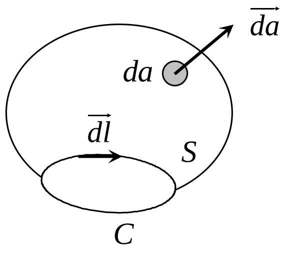

Figure 2

Moreover, by the *Faraday-Henry law*,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6}$$

So, (5) yields

$$\mathcal{E}(t) = -\frac{d}{dt}\int_S \vec{B} \cdot \overrightarrow{da} = -\frac{d}{dt}\,\Phi_m(t) \tag{7}$$

where

$$\Phi_m(t) = \int_S \vec{B}(\vec{r}, t) \cdot \overrightarrow{da}$$

is the *magnetic flux* through C at time t. As commented in [1], relation (7) expresses a genuine physical law, not a mere consequence of the definition of the emf.

4. Motional emf due to a static magnetic field

Let $C(t)$ be a conducting loop inside a static magnetic field $\vec{B}(\vec{r})$ (Fig. 3). The time dependence of C indicates a motion and/or a deformation of the loop over time. We will show that the emf of C at time t is given by the expression

$$\mathcal{E}(t) = \mathcal{E}_m(t) = \oint_{C(t)} [\vec{\upsilon}(\vec{r}) \times \vec{B}(\vec{r})] \cdot \overrightarrow{dl} \tag{8}$$

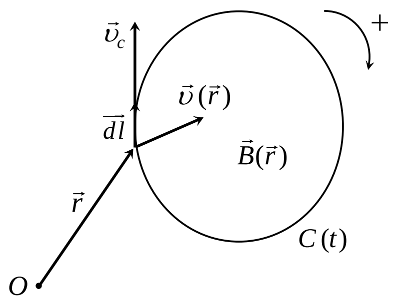

Figure 3

Let q be a mobile charge (say, a conventionally *positive* free electron) located at the position $\vec{r}$ (relative to our coordinate system) of the loop element $\overrightarrow{dl}$ at time t. As in Sec. 1, we denote the velocity of $\overrightarrow{dl}$ with respect to our frame of reference by $\vec{\upsilon}(\vec{r})$, the velocity of q along C by $\vec{\upsilon}_c$, and the total velocity of q relative to our frame by $\vec{\upsilon}_{tot} = \vec{\upsilon} + \vec{\upsilon}_c$.

Since there is no electric field in the region of interest,

$$\mathcal{E}_e(t) \equiv \oint_C \vec{E}(\vec{r},t)\cdot\overrightarrow{dl} = 0 \quad \text{and} \quad \mathcal{E}_{em}(t) = \mathcal{E}_m(t) \tag{9}$$

Also, if $\vec{f}_{app}$ is the applied force per unit charge at the position of q, at time t,

$$\mathcal{E}_{app}(t) = \oint_{C(t)} \vec{f}_{app}(\vec{r},t)\cdot\overrightarrow{dl} \tag{10}$$

The role of the applied force is to keep the current flowing. This will happen for as long as the loop C is moving or/and deforming, so that $\vec{\upsilon}(\vec{r})$ is not identically zero for all t. Why is an external force needed to keep C moving or deforming? Let us carefully analyze the situation.

The magnetic force on q is

$$\vec{F}_m = q\,(\vec{\upsilon}_{tot}\times\vec{B}) \quad \text{so that} \quad \vec{f}_m = \vec{\upsilon}_{tot}\times\vec{B}\ .$$

Now, imagine a temporary, local 3-dimensional rectangular system of axes (x, y, z) at the location $\vec{r}$ of q at time t. We assume, without loss of generality, that the z-axis is in the direction of $\overrightarrow{dl}$. (The orientation of the mutually perpendicular x and y-axes on the plane normal to the z-axis may be chosen arbitrarily.) Then we may write

$$\vec{f}_m = \vec{f}_{m,x} + \vec{f}_{m,y} + \vec{f}_{m,z} \equiv \vec{f}_c + \vec{f}_\perp$$

where $\vec{f}_c = \vec{f}_{m,z}$ is the component of the magnetic force *along* the loop (i.e., in a direction parallel to $\overrightarrow{dl}$) while $\vec{f}_\perp = \vec{f}_{m,x} + \vec{f}_{m,y}$ is the component *normal* to the loop (thus to $\overrightarrow{dl}$).

In a steady-state situation (steady current flow) $\vec{f}_c$ is counterbalanced by the resistive force that opposes charge motion along C (as mentioned before, this latter force does not contribute to the emf). However, to counterbalance the normal component $\vec{f}_\perp$ some external action (say, by our hand that moves or deforms the loop C) is needed in order for C to keep moving or deforming. This is precisely what the applied force $\vec{f}_{app}$ does. Clearly, this force must be *normal* to C at each point of the loop. From (10) we then conclude that

$$\mathcal{E}_{app}(t) = 0 \,.$$

Combining this with (3), (4) and (9), we finally verify the validity of (8).

It can be shown [1,3] directly from (8) that

$$\mathcal{E}(t) = -\frac{d}{dt}\,\Phi_m(t) \tag{11}$$

where $\Phi_m(t)$ is the magnetic flux through C at time t. This *looks like* (7) for a fixed geometrical loop in a time-dependent e/m field, although the origins of the two relations are different. Indeed, equation (11) is a direct consequence of the definition of the emf and may be derived from (8) essentially by mathematical manipulation (see, e.g., the Appendix in [1]). On the contrary, to derive (7) the Faraday-Henry law (6) was used. This is an *experimental* law, hence so is the expression (7) for the emf. In other words, relation (7) is not a mere mathematical consequence of the definition of the emf.

5. An example

Consider a metal bar (ab) of length h, sliding parallel to itself with constant speed υ on two parallel rails that form part of a U-shaped wire, as shown in Fig. 4. A *uniform* magnetic field $\vec{B}$, pointing into the page, fills the entire region. A circuit $C(t)$ of variable size is formed by the rectangular loop ($abcda$).

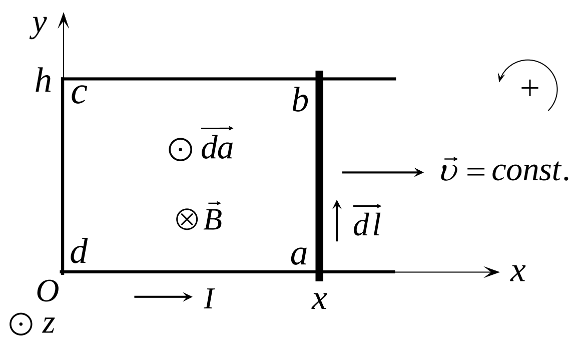

Figure 4

In Fig. 4, the z-axis is normal to the plane of the wire and directed toward the reader. We call $\overrightarrow{da}$ an infinitesimal normal vector representing an element of the plane surface bounded by the wire (this vector is directed toward the reader, consistently with the chosen counterclockwise direction of traversing the loop C). If $\hat{u}_z$ is the unit vector on the z-axis, then the field and the surface element are written, respectively, as $\vec{B}=-B\hat{u}_z$ (where $B=|\vec{B}|=const.$) and $\overrightarrow{da}=(da)\hat{u}_z$.

The balance of forces is shown in Fig. 5 (by $\vec{f}_r$ we denote the resistive force per unit charge, which does not contribute to the emf). Note that this diagram concerns only the *moving* part (*ab*) of the circuit, since it is in this part only that the velocity $\vec{\upsilon}$ and the applied force $\vec{f}_{app}$ are nonzero.

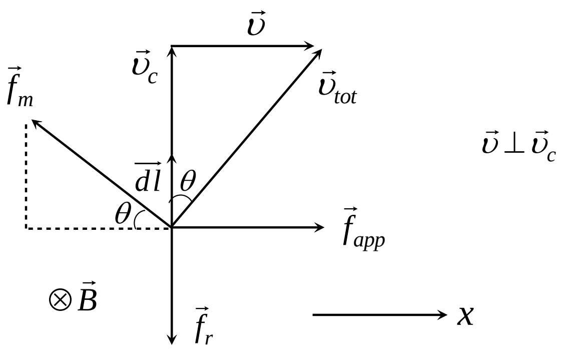

Figure 5

The emf of the circuit at time t is, according to (8),

$$\mathcal{E}(t) = \oint_{C(t)} (\vec{\upsilon}\times\vec{B})\cdot\overrightarrow{dl} = \int_a^b \upsilon B\,dl = \upsilon B\int_a^b dl = \upsilon B h \ .$$

Alternatively, the magnetic flux through C is

$$\Phi_m(t) = \int_{S(t)} \vec{B}\cdot\overrightarrow{da} = -\int_{S(t)} B\,da = -B\int_{S(t)} da = -Bhx$$

(where x is the momentary position of the bar at time t) so that, by (11),

$$\mathcal{E}(t) = -\frac{d}{dt}\Phi_m(t) = Bh\frac{dx}{dt} = Bh\upsilon \ .$$

Now, the role of the applied force is to counterbalance the x-component of the magnetic force in order that the bar may move at constant speed in the x direction. Thus,

$$f_{app} = f_m\cos\theta = \upsilon_{tot}B\cos\theta = B\upsilon_c \ .$$

We note that, although f_{app} depends on the speed υ_c of a mobile charge along the bar, the associated part of the emf is itself independent of υ_c ! Specifically, as argued in

Sec. 4, $\mathcal{E}_{app}(t)=0$. On the other hand, in this particular example the work w of f_{app} for a complete tour around the circuit is equal to the total emf (cf. [2]): $w=\mathcal{E}=Bhv$. This equality, however, is accidental and does not reflect a more general relation between the work per unit charge and the emf. (Another such "accidental" case is the battery-resistance circuit [1-3].)

6. Summary

This article is an addendum to our study of the concept of the electromotive force (emf), as this concept was pedagogically approached in previous publications [1,2]. We have focused on some particular aspects of the subject that we felt are important enough to merit further discussion. Let us review them:

1. For a conducting loop C inside an e/m field, we explained why the emf of C does not depend on the conductivity of the loop. As "obvious" as this statement may seem, one still needs to justify it physically and to demonstrate its consistency with Ohm's law.

2. We expressed the Faraday-Henry law in terms of the emf of a closed conducting curve inside a time-dependent e/m field.

3. We studied the case of motional emf in some detail (see also [2-5]). Particularly important is the role of the applied force in this case. In addition to analyzing this role and, in the process, deriving an explicit expression for the emf, we explained why the physics of the situation is different from that of the Faraday-Henry law, despite the similar-looking forms of the emf in the two cases. Of course, as Relativity has shown, this similarity is anything but coincidental!

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Some remarks on the charging capacitor problem

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Abstract

The charging capacitor is the standard textbook and classroom example for explaining the concept of the so-called Maxwell displacement current. A certain aspect of the problem, however, is often overlooked. It concerns the conditions for satisfaction of the Faraday-Henry law inside the capacitor. Expressions for the electromagnetic field are derived that properly satisfy all four of Maxwell's equations in that region.

1. Introduction

The charging capacitor is the standard paradigm used in intermediate-level Physics courses, textbooks and articles to demonstrate the significance of the Maxwell "displacement current" (see, e.g., [1-7]). The point is correctly made that, without this "current" term, the static Ampère's law would be incomplete with regard to explaining the conservation of charge as well as the existence of electromagnetic radiation. Also, the line integral of the magnetic field around a closed curve would be an ill-defined concept.

There is, however, a certain subtlety of the situation which is often passed by. It concerns the Faraday-Henry law both inside and outside the capacitor. The purpose of this short note is to point out the need for a more careful examination of the satisfaction of this law in the former region, i.e., in the interior of the capacitor. We will seek expressions for the electromagnetic field that properly satisfy the entire set of Maxwell's equations; in particular, the Faraday-Henry law as well as the Ampère-Maxwell law.

2. The standard approach to the charging capacitor problem

We consider a parallel-plate capacitor with circular plates of radius a, thus of area $A=\pi a^2$. The space in between the plates is assumed to be empty of matter. The capacitor is being charged by a time-dependent current $I(t)$ flowing in the $+z$-direction. The z-axis is perpendicular to the plates (the latter are therefore parallel to the xy-plane) and passes through their centers, as seen in Fig. 1 (by $\hat{u}_z$ we denote the unit vector in the $+z$ direction):

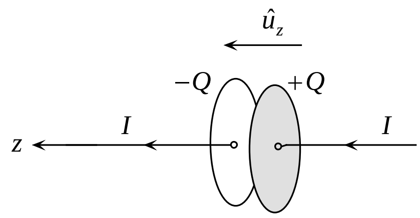

Figure 1: A current I charging a parallel-plate capacitor

The capacitor is being charged at a rate $dQ/dt=I(t)$, where $+Q(t)$ is the charge on the right plate (as seen in the figure) at time t. If $\sigma(t)=Q(t)/\pi a^2=Q(t)/A$ is the surface charge density on the right plate, then the time derivative of σ is given by

$$\sigma'(t) = \frac{I(t)}{A} \tag{1}$$

We assume that the plate separation is very small compared to the radius a, so that the electromagnetic (e/m) field inside the capacitor is practically independent of z, although it *does* depend on the normal distance ρ from the z-axis. (We will not be concerned with edge effects, thus we will restrict out attention to points that are not close to the edges of the plates.) In cylindrical coordinates (ρ,φ,z) the e/m field at any time t will thus only depend on ρ (it will not depend on the angle φ, as follows by the symmetry of the problem).

The magnetic field inside the capacitor is azimuthal, of the form

$$\vec{B} = B(\rho,t)\,\hat{u}_\varphi \,.$$

A standard practice is to assume that the electric field in that area is uniform, of the form

$$\vec{E} = \frac{\sigma(t)}{\varepsilon_0}\,\hat{u}_z \tag{2}$$

while everywhere outside the capacitor the electric field vanishes. With this assumption the magnetic field inside the capacitor is found to be [2,3,6]

$$\vec{B} = \frac{\mu_0 I(t)\rho}{2\pi a^2}\,\hat{u}_\varphi = \frac{\mu_0 I(t)\rho}{2A}\,\hat{u}_\varphi \qquad (3)$$

Expressions (2) and (3) must, of course, satisfy the Maxwell system of equations in empty space, which system we choose to write in the form [1,8]

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E} = 0 \qquad &(c)\quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B} = 0 \qquad &(d)\quad \vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \qquad (4)$$

By using cylindrical coordinates and by taking (1) into account, it is not hard to show that (2) and (3) satisfy three of Eqs. (4), namely, (*a*), (*b*) and (*d*). This is not the case with the Faraday-Henry law (4*c*), however, since by (2) and (3) we find that

$$\vec{\nabla}\times\vec{E} = 0\,,$$

while

$$\frac{\partial\vec{B}}{\partial t} = \frac{\mu_0 I'(t)\rho}{2A}\,\hat{u}_\varphi\,.$$

An exception occurs if the current I is constant in time, i.e., if the capacitor is being charged at a constant rate, so that $I'(t)=0$ (this is, e.g., the assumption made in [2]). But, for a current $I(t)$ with arbitrary time dependence, the pair of fields (2) and (3) does not satisfy the third Maxwell equation.

3. A more general formula for the e/m field inside the capacitor

To remedy the situation and restore the validity of the full set of Maxwell's equations in the interior of the capacitor, we must somehow correct the expressions (2) and (3) for the e/m field. To this end, we make use of the following *Ansatz*:

$$\begin{aligned} &\vec{E} = \left(\frac{\sigma(t)}{\varepsilon_0} + f(\rho,t)\right)\hat{u}_z\,, \\ &\vec{B} = \left(\frac{\mu_0 I(t)\rho}{2A} + g(\rho,t)\right)\hat{u}_\varphi\,; \\ &\sigma'(t) = I(t)/A \end{aligned} \qquad (5)$$

where $f(\rho,t)$ and $g(\rho,t)$ are functions to be determined consistently with the given current function $I(t)$ and for given initial conditions. It is easy to check that the solutions (5) automatically satisfy the first two Maxwell equations (4*a*) and (4*b*). By the Faraday-Henry law (4*c*) and the Ampère-Maxwell law (4*d*) we get the following system of partial differential equations:

$$\begin{aligned} &\frac{\partial f}{\partial\rho} = \frac{\partial g}{\partial t} + \frac{\mu_0 I'(t)\rho}{2A} \qquad &(a) \\ &\frac{\partial(\rho g)}{\partial\rho} = \varepsilon_0\mu_0\frac{\partial(\rho f)}{\partial t} \qquad &(b) \end{aligned} \qquad (6)$$

Note in particular that the "classical" solution with $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$ is possible only if $I'(t)=0 \Leftrightarrow I$=constant in time (i.e., if the capacitor is being charged at a constant rate), as mentioned earlier.

As a special case, let us assume that the functions f and g are time-independent, i.e., $\partial f/\partial t = \partial g/\partial t = 0 \Leftrightarrow f=f(\rho)$, $g=g(\rho)$. From (6*a*) we get:

$$f(\rho) = \frac{\mu_0 I'(t)\rho^2}{4A}\,.$$

This can only be valid if $I'(t)$=constant $\Leftrightarrow I''(t)=0$. On the other hand, (6*b*) yields: ρg=constant $\equiv \lambda \Leftrightarrow g(\rho)=\lambda/\rho$. In order for $g(\rho)$ to be finite for $\rho=0$, we must set $\lambda=0$, so that $g(\rho)\equiv 0$. The solution (5) for the e/m field inside the capacitor is then written:

$$\begin{aligned} &\vec{E} = \left(\frac{\sigma(t)}{\varepsilon_0} + \frac{\mu_0 I'(t)\rho^2}{4A}\right)\hat{u}_z\,, \\ &\vec{B} = \frac{\mu_0 I(t)\rho}{2A}\,\hat{u}_\varphi\,; \\ &I''(t) = 0\,,\quad \sigma'(t) = I(t)/A \end{aligned} \qquad (7)$$

This formula preserves the familiar expression (3) for the magnetic field but corrects Eq. (2) for the electric field in order that the Faraday-Henry law be satisfied.

4. Summary

The purpose of this note was to point out the need to revisit the problem of the charging capacitor and to carefully examine the expressions for the e/m field in the interior of this system. As was noted, the standard formulas assumed for this field, tailor-made to satisfy the Ampère-Maxwell law, fail to satisfy the Faraday-Henry law except in the special case where the capacitor is being charged at a constant rate. We have derived a general expression for the e/m field that satisfies the full set of Maxwell's equations for arbitrary charging rate of the system. This result reduces to the familiar set of equations in the case of a constant charging rate.

Analogous corrections need to be made to the standard expressions for the e/m field in the *exterior* of the capacitor. This will be the subject of a subsequent paper.

Acknowledgement

I thank Aristidis N. Magoulas for a number of helpful discussions on this problem.

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On solving Maxwell’s equations for a charging capacitor

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Abstract. The charging capacitor is used as a standard paradigm for illustrating the concept of the Maxwell “displacement current”. A certain aspect of the problem, however, is often overlooked. It concerns the conditions for satisfaction of the Faraday-Henry law both in the interior and the exterior of the capacitor. In this article the situation is analyzed and a mathematical process is described for obtaining expressions for the electromagnetic field that satisfy the full set of Maxwell’s equations both inside and outside the capacitor.

Keywords: Maxwell equations, Faraday-Henry law, displacement current, capacitor

PACS: 03.50.De, 41.20.-q, 01.55.+b, 01.40.Fk

1. Introduction

The charging capacitor is the standard paradigm used in intermediate-level Physics courses, textbooks and articles to demonstrate the significance of the Maxwell “displacement current” (see, e.g., [1-7]). The point is correctly made that, without this “current” term, the static Ampère’s law would be incomplete with regard to explaining the conservation of charge as well as the existence of electromagnetic radiation. Also, the line integral of the magnetic field around a closed curve would be an ill-defined concept.

There is, however, a certain subtlety of the situation which is often passed by. It concerns the satisfaction of the Faraday-Henry law both inside and outside the capacitor. Indeed, although care is taken to ensure that the expressions used for the electromagnetic (e/m) field satisfy the Ampère-Maxwell law, no such care is exercised with regard to the Faraday-Henry law. As it turns out, the usual formulas for the e/m field satisfy this latter law only in the special case where the capacitor is being charged at a constant rate. But, if the current responsible for charging the capacitor is time-dependent, this will also be the case with the magnetic field outside the capacitor. This, in turn, implies the existence of an “induced” electric field in that region, contrary to the usual assertion that the electric field outside the capacitor is zero. Moreover, the time dependence of the magnetic field inside the capacitor is not compatible with the assumption that the electric field in that region is uniform, as the case would be in a static situation.

The purpose of this article is to exhibit the theoretical inconsistencies inherent in the “classical” treatment of the charging capacitor problem and to describe a mathematical process

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for deriving expressions for the e/m field that satisfy the full set of the Maxwell equations (including, of course, the Faraday-Henry law) both inside and outside the capacitor.

After a preliminary discussion of the concept of the electric current through a loop (Section 2), the standard "textbook" approach to the charging-capacitor example in connection with the concept of the displacement current is presented in Section 3. New and more general solutions of the Maxwell system of equations in the interior and the exterior of the capacitor are then derived in Sections 4 and 5, respectively.

2. The current through a loop

Before we proceed to write the Ampère-Maxwell law in its integral form, we must carefully define the concept of the *total current through a loop C* (where by "loop" we mean a closed curve in space).

Proposition. Consider a region R of space within which the distribution of charge, expressed by the volume charge density ρ, is time-independent ($\partial\rho/\partial t=0$). Let C be an oriented loop in R, and let S be any open surface in R bordered by C and oriented accordingly. We define the total current through C as the surface integral of the current density $\vec{J}$ over S:

$$I_{in} = \int_S \vec{J} \cdot \overrightarrow{da} \qquad (1)$$

Then, the quantity I_{in} has a well-defined value independent of the particular choice of S (that is, I_{in} is the same for all open surfaces S bounded by C).

Proof. By the equation of continuity for the electric charge (see, e.g., [8], Chap. 6) and by the fact that the charge density ρ inside the region R is static ($\partial\rho/\partial t=0$), we have that $\vec{\nabla} \cdot \vec{J} = 0$. Therefore, within this region of space the current density has the properties of a solenoidal field. In particular, the value of the surface integral of $\vec{J}$ will be the same for all open surfaces S sharing a common border C.

As an example, let us consider a circuit carrying a time-dependent current $I(t)$. If the circuit does not contain a capacitor, no charge is piling up at any point and the charge density at any elementary segment of the circuit is constant in time. Moreover, at each instant t, the current I is constant along the circuit, its value changing only with time. Now, if C is a loop encircling some section the circuit, as shown in Fig. 1, then, at each instant t, the same current $I(t)$ will pass through any open surface S bordered by C. Thus, the integral in (1) is well defined for all t, assuming the same value $I_{in}=I(t)$ for all S.

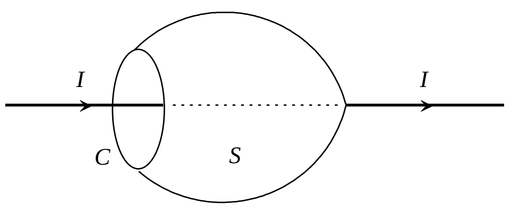

Figure 1

Things change if the circuit contains a capacitor which is charging or discharging. It is then no longer true that the current $I(t)$ is constant along the circuit; indeed, $I(t)$ is zero inside the capacitor and nonzero outside. Thus, the value of the integral in (1) depends on whether the surface S does or does not contain points belonging to the interior of the capacitor.

3. Maxwell displacement current in a charging capacitor

Figure 2 shows a simple circuit containing a capacitor that is being charged by a time-dependent current $I(t)$. At time t, the plates of the capacitor, each of area A, carry charges $\pm Q(t)$.

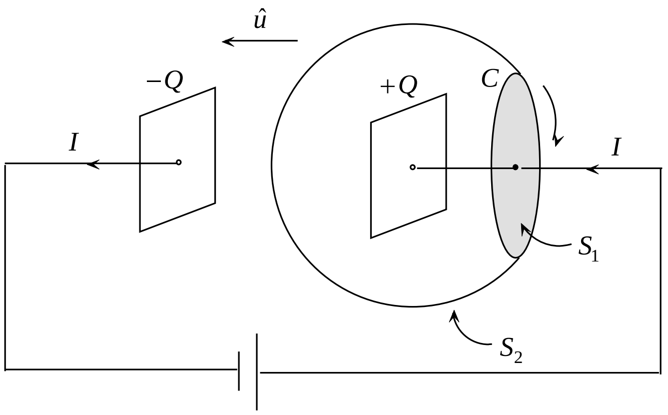

Figure 2

Assume that we encircle the current I by an imaginary plane loop C parallel to the positive plate and oriented in accordance with the "right-hand rule", consistently with the direction of I (this direction is indicated by the unit vector $\hat{u}$). The "current through C" is here an ill-defined notion since the value of the integral in Eq. (1) is $I_{in}=I$ for the flat surface S_1 and $I_{in}=0$ for the curved surface S_2 (Fig. 2). This, in turn, implies that Ampère's law of magnetostatics [1-4,8] cannot be valid in this case, given that, according to this law, the integral of the magnetic field $\vec{B}$ along the loop C, equal to $\mu_0 I_{in}$, would not be uniquely defined but would depend on the surface S bounded by C.

Maxwell restored the single-valuedness of the closed line integral of $\vec{B}$ by introducing the so-called *displacement current*, which is essentially the rate of change of a time-dependent electric field:

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Leftrightarrow \quad I_d = \int_S \vec{J}_d \cdot \overrightarrow{da} = \varepsilon_0 \int_S \frac{\partial \vec{E}}{\partial t} \cdot \overrightarrow{da} \qquad (2)$$

The *Ampère-Maxwell law* reads:

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$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \quad \Leftrightarrow$$
$$\oint_C \vec{B}\cdot\vec{dl}=\mu_0 I_{in}+\varepsilon_0\mu_0\int_S\frac{\partial\vec{E}}{\partial t}\cdot\vec{da}\equiv\mu_0(I+I_d)_{in} \tag{3}$$

where I_{in} is given by Eq. (1).

Now, the standard “textbook” approach to the charging capacitor problem goes as follows: Outside the capacitor the electric field vanishes everywhere, while inside the capacitor the electric field is uniform – albeit time-dependent – and has the static-field-like form

$$\vec{E}=\frac{\sigma(t)}{\varepsilon_0}\hat{u}=\frac{Q(t)}{\varepsilon_0 A}\hat{u} \tag{4}$$

where $\sigma(t)=Q(t)/A$ is the surface charge density on the positive plate at time t. This density is related to the current I, which charges the capacitor, by

$$\sigma'(t)=\frac{Q'(t)}{A}=\frac{I(t)}{A} \tag{5}$$

(the prime indicates differentiation with respect to t). Thus, inside the capacitor,

$$\frac{\partial\vec{E}}{\partial t}=\frac{\sigma'(t)}{\varepsilon_0}\hat{u}=\frac{I(t)}{\varepsilon_0 A}\hat{u} \tag{6}$$

Outside the capacitor the time derivative of the electric field vanishes everywhere and, therefore, so does the displacement current.

Now, on the flat surface S_1 the total current through C is $(I+I_d)_{in}=I+0=I(t)$. The Ampère-Maxwell law (3) then yields:

$$\int_C \vec{B}\cdot\vec{dl}=\mu_0 I(t) \tag{7}$$

On the curved surface S_2, the total current through C is $(I+I_d)_{in}=0+I_{d,in}=I_{d,in}$, where the quantity on the right assumes a nonzero value only for the portion S_2' of S_2 which lies inside the capacitor. This quantity is equal to

$$I_{d,in}=\varepsilon_0\int_{S_2'}\frac{\partial\vec{E}}{\partial t}\cdot\vec{da}=\frac{I(t)}{A}\int_{S_2'}\hat{u}\cdot\vec{da} \tag{8}$$

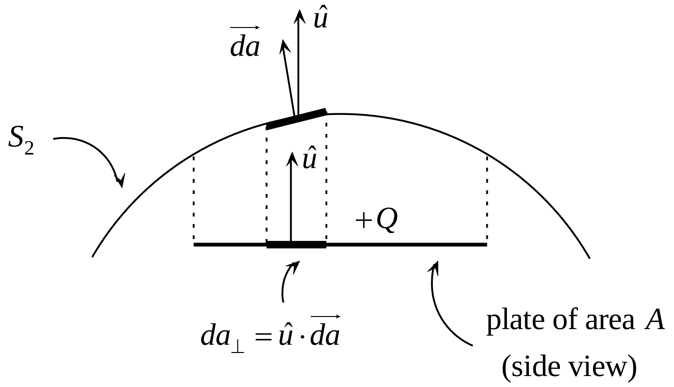

Figure 3

The dot product in the integral on the right of (8) represents the projection of the surface element $\vec{da}$ onto the axis defined by the unit vector $\hat{u}$ (see Fig. 3). This is equal to the projection $da_\perp$ of an elementary area da of S_2' onto the flat surface of the plate of the capacitor. Eventually, the integral on the right of (8) equals the total area A of the plate. Hence, $I_{d,in}=I(t)$ and, given that $I_{in}=0$ on S_2, the Ampère-Maxwell law (3) again yields the result (7).

So, everything works fine with regard to the Ampère-Maxwell law, but there is one law we have forgotten so far; namely, the *Faraday-Henry law*! According to that law, a time-changing magnetic field is always accompanied by an electric field (or, as is often said, "induces" an electric field). So, the electric field outside the capacitor cannot be zero, as claimed previously, given that the time-dependent current $I(t)$ is expected to generate a time-dependent magnetic field. For a similar reason, the electric field inside the capacitor cannot have the static-field-like form (4) (there must also be a contribution from the rate of change of the magnetic field between the plates).

An exception occurs if the current I which charges the capacitor is constant in time, since in this case the magnetic field will be static everywhere. This is actually the assumption silently or explicitly made in many textbooks (see, e.g., [2], Chap. 21). Physically this means that the capacitor is being charged at a constant rate. But, in the general case where $I(t)\neq$constant, the preceding discussion regarding the charging capacitor problem needs to be significantly revised in order to take into account the entire set of the Maxwell equations; in particular, the Ampère-Maxwell law as well as the Faraday-Henry law.

4. The Maxwell equations inside the capacitor

We consider a parallel-plate capacitor with circular plates of radius a, thus of area $A=\pi a^2$. The space in between the plates is assumed to be empty of matter. The capacitor is being charged by a time-dependent current $I(t)$ flowing in the $+z$ direction. The z-axis is perpendicular to the plates (the latter are therefore parallel to the xy-plane) and passes through their centers, as seen in Fig. 4 (by $\hat{u}_z$ we denote the unit vector in the $+z$ direction).

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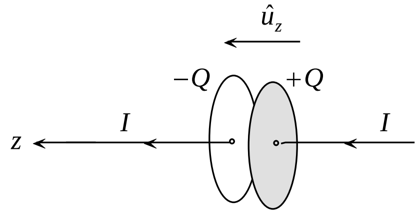

Figure 4

The capacitor is being charged at a rate $dQ/dt=I(t)$, where $+Q(t)$ is the charge on the right plate (as seen in the figure) at time t. If $\sigma(t)=Q(t)/\pi a^2=Q(t)/A$ is the surface charge density on the right plate, then the time derivative of σ is given by Eq. (5).

We assume that the plate separation is very small compared to the radius a, so that the electromagnetic (e/m) field inside the capacitor is practically independent of z, although it *does* depend on the normal distance ρ from the z-axis. (We will not be concerned with edge effects, thus we will restrict out attention to points that are not too close to the edges of the plates.) In cylindrical coordinates (ρ, φ, z) the magnitude of the e/m field at any time t will thus only depend on ρ (it will not depend on the angle φ, as follows by the symmetry of the problem).

We assume that the positive and the negative plate of the capacitor of Fig. 4 are centered at $z=0$ and $z=d$, respectively, on the z-axis, where, as mentioned above, the plate separation d is much smaller than the radius a of the plates. The interior of the capacitor is then the region of space with $0 \le \rho < a$ and $0 < z < d$.

The magnetic field inside the capacitor is azimuthal, of the form $\vec{B} = B(\rho, t)\hat{u}_\varphi$. As noted in Sec. 3, a standard practice is to assume that, at all t, the electric field in this region is uniform, of the form

$$\vec{E} = \frac{\sigma(t)}{\varepsilon_0}\,\hat{u}_z \tag{9}$$

while everywhere outside the capacitor the electric field vanishes. With this assumption the magnetic field inside the capacitor is found to be [2,3,6]

$$\vec{B} = \frac{\mu_0 I(t)\rho}{2\pi a^2}\,\hat{u}_\varphi = \frac{\mu_0 I(t)\rho}{2A}\,\hat{u}_\varphi \tag{10}$$

Expressions (9) and (10) must, of course, satisfy the Maxwell system of equations in empty space, which system we write in the form [1,8]

$$\begin{array}{llll} (a) & \vec{\nabla}\cdot\vec{E} = 0 & (c) & \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ (b) & \vec{\nabla}\cdot\vec{B} = 0 & (d) & \vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\dfrac{\partial \vec{E}}{\partial t} \end{array} \tag{11}$$

By using cylindrical coordinates (see Appendix) and by taking into account that $\sigma'(t)=I(t)/A$ [Eq. (5)], it is not hard to show that (9) and (10) satisfy three of Eqs. (11), namely, (*a*), (*b*) and (*d*). This is not the case with the Faraday-Henry law (11*c*), however, since by (9) and (10) we find that $\vec{\nabla}\times\vec{E}=0$, while

$$\frac{\partial\vec{B}}{\partial t}=\frac{\mu_0 I'(t)\rho}{2A}\,\hat{u}_\varphi \;.$$

An exception occurs if the current I is constant in time, i.e., if the capacitor is being charged at a constant rate, so that $I'(t)=0$. But, for a current $I(t)$ with arbitrary time dependence, the pair of fields (9) and (10) does not satisfy the third Maxwell equation.

To remedy the situation and restore the validity of the full set of Maxwell's equations in the interior of the capacitor, we must somehow correct the expressions (9) and (10) for the e/m field. To this end, we employ the following *Ansatz*:

$$\vec{E}=\left(\frac{\sigma(t)}{\varepsilon_0}+f(\rho,t)\right)\hat{u}_z\;,\quad \vec{B}=\left(\frac{\mu_0 I(t)\rho}{2A}+g(\rho,t)\right)\hat{u}_\varphi\;; \qquad \sigma'(t)=I(t)/A \tag{12}$$

where $f(\rho,t)$ and $g(\rho,t)$ are functions to be determined consistently with the given current function $I(t)$ and the given initial conditions. It is easy to check that the solutions (12) automatically satisfy the first two Maxwell equations (11*a*) and (11*b*). By the Faraday-Henry law (11*c*) and the Ampère-Maxwell law (11*d*) we get the following system of partial differential equations:

$$\begin{aligned} &\frac{\partial f}{\partial\rho}=\frac{\partial g}{\partial t}+\frac{\mu_0 I'(t)\rho}{2A} && (a)\\ &\frac{\partial(\rho g)}{\partial\rho}=\varepsilon_0\mu_0\frac{\partial(\rho f)}{\partial t} && (b)\end{aligned} \tag{13}$$

Note in particular that the "classical" solution with $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$ is possible only if $I'(t)=0 \Leftrightarrow I=$constant in time (i.e., if the capacitor is being charged at a constant rate), as mentioned earlier.

As a special case, let us assume that the functions f and g are time-independent, i.e., $\partial f/\partial t=\partial g/\partial t=0 \Leftrightarrow f=f(\rho)$, $g=g(\rho)$. From (13*a*) we get (ignoring an arbitrary constant):

$$f(\rho)=\frac{\mu_0 I'(t)\rho^2}{4A}\;.$$

This can only be valid if $I'(t)=$constant $\Leftrightarrow I''(t)=0$. On the other hand, (13*b*) yields: $\rho g=$constant $\equiv\lambda \Leftrightarrow g(\rho)=\lambda/\rho$. In order for $g(\rho)$ to be finite for $\rho=0$, we must set $\lambda=0$, so that $g(\rho)\equiv 0$. The solution (12) for the e/m field inside the capacitor is then written:

$$\vec{E}=\left(\frac{\sigma(t)}{\varepsilon_0}+\frac{\mu_0 I'(t)\rho^2}{4A}\right)\hat{u}_z\ ,\quad \vec{B}=\frac{\mu_0 I(t)\rho}{2A}\,\hat{u}_\varphi\ ;$$
$$I''(t)=0\ ,\quad \sigma'(t)=I(t)/A \tag{14}$$

We notice that, since $I''(t)=0$, Eq. (6) is still valid and the displacement current inside the capacitor is again given by $I_d=I(t)$. What *is* different here is the correction to the electric field in order for the Faraday-Henry law to be satisfied.

5. The Maxwell equations outside the capacitor

We recall that the positive and the negative plate of the capacitor of Fig. 4 are centered at $z=0$ and $z=d$, respectively, on the z-axis, where the plate separation d is much smaller than the radius a of the plates. The space exterior to the capacitor consists of points with $\rho>0$ and $z\notin(0,d)$, as well as points with $\rho>a$ and $0<z<d$. (In the former case we exclude points on the z-axis, with $\rho=0$, to ensure the finiteness of our solutions in that region.)

The e/m field outside the capacitor is usually described mathematically by the equations [2,3,6]

$$\vec{E}=0\ ,\quad \vec{B}=\frac{\mu_0 I(t)}{2\pi\rho}\,\hat{u}_\varphi \tag{15}$$

As the case is with the standard solutions in the interior of the capacitor, the solutions (15) fail to satisfy the Faraday-Henry law (11c) (although they do satisfy the remaining three Maxwell equations), since $\vec{\nabla}\times\vec{E}=0$ while

$$\frac{\partial\vec{B}}{\partial t}=\frac{\mu_0 I'(t)}{2\pi\rho}\,\hat{u}_\varphi\ .$$

As before, an exception occurs if the current I is constant in time, i.e., if the capacitor is being charged at a constant rate, so that $I'(t)=0$.

To find more general solutions that satisfy the entire set of the Maxwell equations, we work as in the previous section. Thus, we assume the following general form of the e/m field everywhere outside the capacitor:

$$\vec{E}=f(\rho,t)\,\hat{u}_z\ ,\quad \vec{B}=\left(\frac{\mu_0 I(t)}{2\pi\rho}+g(\rho,t)\right)\hat{u}_\varphi \tag{16}$$

where f and g are functions to be determined consistently with the given current function $I(t)$. The solutions (16) automatically satisfy the first two Maxwell equations (11a) and (11b). By Eqs. (11c) and (11d) we get the following system of partial differential equations:

$$\begin{aligned} \frac{\partial f}{\partial \rho} &= \frac{\partial g}{\partial t} + \frac{\mu_0 I'(t)}{2\pi\rho} && (a) \\ \frac{\partial(\rho g)}{\partial \rho} &= \varepsilon_0 \mu_0 \frac{\partial(\rho f)}{\partial t} && (b) \end{aligned} \qquad (17)$$

Again, the usual solution with $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$ is possible only if $I'(t)=0$, i.e., if the capacitor is being charged at a constant rate.

As a special case, let us assume that the functions f and g are time-independent, i.e., $f=f(\rho)$, $g=g(\rho)$. From (17*a*) we get:

$$f(\rho) = \frac{\mu_0 I'(t)}{2\pi} \ln(\kappa\rho)$$

where κ is a positive constant quantity having dimensions of inverse length. This can only be valid if $I'(t)$=constant $\Leftrightarrow I''(t)=0$. On the other hand, (17*b*) yields: ρg=constant $\equiv \lambda \Leftrightarrow g(\rho)=\lambda/\rho$. Since $\rho>0$, by assumption, we could now let $\lambda\neq 0$. For reasons of continuity, however (see below), we set $\lambda=0$, so that $g=0$. The solution (16) for the e/m field outside the capacitor is then written:

$$\begin{aligned} &\vec{E} = \frac{\mu_0 I'(t)}{2\pi} \ln(\kappa\rho)\, \hat{u}_z \ , \quad \vec{B} = \frac{\mu_0 I(t)}{2\pi\rho}\, \hat{u}_\varphi \ ; \\ &I''(t) = 0 \end{aligned} \qquad (18)$$

Note, in particular, that the magnetic field in the strip $0<z<d$ is continuous for $\rho=a$, since the expression for $\vec{B}$ in (18) matches the corresponding expression in (14) upon substituting $\rho=a$ (remember that $A=\pi a^2$). No analogous continuity exists, however, for the electric field. Physically, this may be attributed to fringing effects at the edges of the plates.

6. Summary

The purpose of this article is to point out the need to revisit the problem of the charging capacitor, as this is discussed in connection with the Maxwell displacement current, and to carefully examine the expressions for the e/m field both in the interior and the exterior of this system. As was noted, the standard formulas assumed for this field, tailor-made to satisfy the Ampère-Maxwell law, fail to satisfy the Faraday-Henry law except in the special case where the capacitor is being charged at a constant rate. We have derived general expressions for the e/m field that satisfy the full set of Maxwell's equations for arbitrary charging rate of the system. These results may reduce to the familiar set of equations in the case of a constant charging rate.

Note

This article is an extensively revised and expanded version of an article published previously in letter form [9]. In particular, the results contained in Sec. 5 of this article are new.

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Appendix: Vector operators in cylindrical coordinates

Let $\vec{A}$ be a vector field, expressed in cylindrical coordinates (ρ, φ, z) as

$$\vec{A} = A_\rho(\rho,\varphi,z)\,\hat{u}_\rho + A_\varphi(\rho,\varphi,z)\,\hat{u}_\varphi + A_z(\rho,\varphi,z)\,\hat{u}_z \ .$$

The *div* and the *rot* of this field, in this system of coordinates, are written respectively as follows:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z} \ ,$$

$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{u}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{u}_\varphi + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial\varphi}\right)\hat{u}_z \ .$$

In particular, if the vector field is of the form $\vec{A} = A_\varphi(\rho)\,\hat{u}_\varphi + A_z(\rho)\,\hat{u}_z$, then $\vec{\nabla}\cdot\vec{A} = 0$.

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Approximate solutions of Maxwell's equations for a charging capacitor

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Abstract. In previous articles we derived a system of partial differential equations by means of which one may obtain expressions for the electromagnetic field in the interior and the exterior of a charging capacitor. In the present article a recursive process is described for finding solutions of this system in power-series form with respect to time. This allows one to find approximate solutions of Maxwell's equations in a number of situations of physical interest.

Keywords: Maxwell's equations, Faraday's law, charging capacitor

1. Introduction

In previous articles [1,2] we described a mathematical process for finding expressions for the electromagnetic (e/m) field – i.e., solutions of Maxwell's equations – in the interior and the exterior of a charging capacitor. These solutions generalize the "classical" results found in the educational literature of electrodynamics [3-9], which results were noted to not satisfy, in general, the Faraday-Henry law (Maxwell's third equation).

Our method was based on a simple idea: we started with the known (incomplete) solutions and "corrected" them by adding unknown functions to be determined by using the Maxwell system. This led to a system of partial differential equations (PDEs) for these functions, in which system the (generally) time-dependent current that charges the capacitor appears as a sort of parametric function.

In the present article we suggest a mathematical process for obtaining solutions of the aforementioned system of PDEs in the form of power series with respect to time. This allows one to find approximate expressions for the e/m field in certain situations. For example, a slowly varying (thus almost time-independent) current allows for the "classical" solutions given in the literature, while a current that is almost linearly dependent on time (as may be assumed, in general, for any smoothly varying current in a very short time period) allows for new solutions that correct the standard expressions for the electric field while retaining the corresponding expressions for the magnetic field.

It should be noted that, regarding the solutions in the exterior of the capacitor, no retardation effects related to the finite speed of propagation of e/m interactions will concern us here. Indeed, as discussed in Sec. 4, our solutions are valid at points of

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space not far from the capacitor, so that any change in the physical system will be felt "simultaneously" at all points of interest.

2. Solutions of Maxwell's equations inside the capacitor

We consider a parallel-plate capacitor with circular plates of radius a, thus of area $A=\pi a^2$. The space in between the plates is assumed to be empty of matter. The capacitor is being charged by a time-dependent current $I(t)$ flowing in the $+z$ direction (see Fig. 1). The z-axis is perpendicular to the plates (the latter are therefore parallel to the xy-plane) and passes through their centers, as seen in the figure (by $\hat{u}_z$ we denote the unit vector in the $+z$ direction).

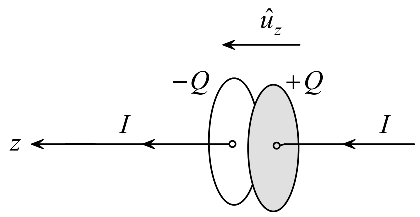

Figure 1

The capacitor is being charged at a rate $dQ/dt=I(t)$, where $+Q(t)$ is the charge on the right plate (as seen in the figure) at time t. If $\sigma(t)=Q(t)/\pi a^2=Q(t)/A$ is the surface charge density on the right plate, then the time derivative of σ is given by

$$\sigma'(t)=\frac{Q'(t)}{A}=\frac{I(t)}{A} \tag{1}$$

We assume that the plate separation is very small compared to the radius a, so that the e/m field inside the capacitor is practically independent of z, although it *does* depend on the normal distance ρ from the z-axis. In cylindrical coordinates (ρ,φ,z) the magnitude of the e/m field at any time t will thus only depend on ρ (due to the symmetry of the problem, this magnitude will not depend on the angle φ).

We assume that the positive and the negative plate of the capacitor of Fig. 1 are centered at $z=0$ and $z=d$, respectively, on the z-axis, where, as mentioned above, the plate separation d is much smaller than the radius a of the plates. The interior of the capacitor is then the region of space with $0\leq\rho<a$ and $0<z<d$.

The magnetic field inside the capacitor is azimuthal, of the form $\vec{B}=B(\rho,t)\hat{u}_\varphi$. A standard practice in the literature is to assume that, at all t, the electric field in this region is uniform, of the form

$$\vec{E}=\frac{\sigma(t)}{\varepsilon_0}\,\hat{u}_z \tag{2}$$

while everywhere outside the capacitor the electric field vanishes. With this assumption the magnetic field inside the capacitor is found to be [4,5,8]

$$\vec{B} = \frac{\mu_0 I(t)\rho}{2\pi a^2}\,\hat{u}_\varphi = \frac{\mu_0 I(t)\rho}{2A}\,\hat{u}_\varphi \tag{3}$$

Expressions (2) and (3) must, of course, satisfy the Maxwell system of equations in empty space, which system we write in the form [3,10]

$$\begin{array}{llll} (a) & \vec{\nabla}\cdot\vec{E}=0 & (c) & \vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t} \\ (b) & \vec{\nabla}\cdot\vec{B}=0 & (d) & \vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\dfrac{\partial\vec{E}}{\partial t} \end{array} \tag{4}$$

By using cylindrical coordinates (see Appendix I) and by taking (1) into account, one may show that (2) and (3) satisfy three of Eqs. (4), namely, (*a*), (*b*) and (*d*). This is not the case with the Faraday-Henry law (4*c*), however, since by (2) and (3) we find that $\vec{\nabla}\times\vec{E}=0$, while

$$\frac{\partial\vec{B}}{\partial t} = \frac{\mu_0 I'(t)\rho}{2A}\,\hat{u}_\varphi\ .$$

An exception occurs if the current *I* is constant in time, i.e., if the capacitor is being charged at a constant rate, so that $I'(t)=0$. This is actually the assumption silently or explicitly made in many textbooks (see, e.g., [4], Chap. 21). But, for a current $I(t)$ with arbitrary time dependence, the pair of fields (2) and (3) does not satisfy the third Maxwell equation.

To remedy the situation and restore the validity of the full set of Maxwell's equations in the interior of the capacitor, we must somehow correct the above expressions for the e/m field. To this end we employ the following *Ansatz*, taking into account Lemma 1 in Appendix II:

$$\begin{aligned} &\vec{E} = \left(\frac{\sigma(t)}{\varepsilon_0} + f(\rho,t)\right)\hat{u}_z\ , \\ &\vec{B} = \left(\frac{\mu_0 I(t)\rho}{2A} + g(\rho,t)\right)\hat{u}_\varphi\ ; \\ &\sigma'(t) = I(t)/A \end{aligned} \tag{5}$$

where $f(\rho,t)$ and $g(\rho,t)$ are functions to be determined consistently with the given current function $I(t)$ and the given initial conditions. It can be checked that the solutions (5) automatically satisfy the first two Maxwell equations (4*a*) and (4*b*). By the Faraday-Henry law (4*c*) and the Ampère-Maxwell law (4*d*) we get the following system of PDEs:

$$\begin{aligned} \frac{\partial f}{\partial \rho} &= \frac{\partial g}{\partial t} + \frac{\mu_0 I'(t)\rho}{2A} \\ \frac{1}{\rho}\frac{\partial(\rho g)}{\partial \rho} &= \varepsilon_0\mu_0 \frac{\partial f}{\partial t} \end{aligned} \tag{6}$$

Note in particular that the "classical" solution with $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$ is possible only if $I'(t)=0$, i.e., if the current I is constant in time, which means that the capacitor is being charged at a constant rate.

The quantity $(1/\rho)\partial(\rho g)/\partial\rho$ in the second equation, having its origin at the expression for $\vec{\nabla}\times\vec{B}$ in cylindrical coordinates, must tend to a finite limit for $\rho\to 0$ in order that the *rot* of the magnetic field be finite at the center of the capacitor. For this to be the case, $\partial(\rho g)/\partial\rho$ must only contain terms of at least first order in ρ. This, in turn, requires that g itself must be of at least first order (i.e., linear with no constant term) in ρ for all t, or else g must be identically zero. We must, therefore, require that

$$g(\rho,t)\to 0 \;\text{ for }\; \rho\to 0 \tag{7}$$

for all t. Keeping this condition in mind, we can rewrite the system (6) in a more symmetric form:

$$\begin{aligned} \frac{\partial f}{\partial \rho} &= \frac{\partial g}{\partial t} + \frac{\mu_0 I'(t)\rho}{2A} \\ \frac{\partial(\rho g)}{\partial \rho} &= \varepsilon_0\mu_0 \frac{\partial(\rho f)}{\partial t} \end{aligned} \tag{8}$$

In principle, one needs to solve the system (8) for a given current $I(t)$ and for given initial conditions. An alternative approach, leading to approximate solutions of various forms, is to expand all functions (i.e., f, g and I) in powers of time, t. We thus write:

$$I(t) = \sum_{n=0}^{\infty} I_n t^n \tag{9a}$$

$$f(\rho,t) = \sum_{n=0}^{\infty} f_n(\rho)\, t^n \tag{9b}$$

$$g(\rho,t) = \sum_{n=0}^{\infty} g_n(\rho)\, t^n \tag{9c}$$

Then, for example,

$$I'(t) = \sum_{n=1}^{\infty} n I_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) I_{n+1} t^n \,, \;\text{ etc.}$$

Obviously, I_n has dimensions of current × (time)$^{-n}$, while f_n and g_n have dimensions of field intensity (electric and magnetic, respectively) × (time)$^{-n}$.

Substituting the series expansions (9) into the system (8), and equating coefficients of similar powers of t on both sides of the ensuing equations, we get a recursion relation in the form of a system of PDEs:

$$
\begin{aligned}
& f_n'(\rho) = (n+1)\left[g_{n+1}(\rho) + \frac{\mu_0 \rho}{2A} I_{n+1} \right] \\
& \left[\rho g_n(\rho)\right]' = (n+1)\, \varepsilon_0 \mu_0 \rho f_{n+1}(\rho)
\end{aligned}
\tag{10}
$$

for n=0,1,2,... All non-vanishing functions $g_n(\rho)$ are required to satisfy the boundary condition (7); i.e., $g_n(\rho)\to 0$ for $\rho\to 0$.

An obvious solution of the system (10) is the trivial solution $f_n(\rho)\equiv 0$ and $g_n(\rho)\equiv 0$ for all n=0,1,2,..., corresponding to $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$. For this to be the case, we must have $I_{n+1}=0$ for all n=0,1,2,..., which means that $I(t)=I_0$=constant (independent of t). This is the case typically treated in the literature, although the condition I=*const.* is usually not stated explicitly.

The simplest nontrivial solution of the problem is found by assuming that f and g are time-independent, i.e., are functions of ρ only. Then, by (9*b*) and (9*c*), $f=f_0(\rho)$ and $g=g_0(\rho)$, while $f_n(\rho)=0$ and $g_n(\rho)=0$ for $n>0$. The system (10) for $n=0$ gives

$$
f_0'(\rho) = \frac{\mu_0 I_1 \rho}{2A} \quad \text{and} \quad \left[\rho g_0(\rho)\right]' = 0
$$

with solutions

$$
f_0(\rho) = \frac{\mu_0 I_1 \rho^2}{4A} + C \quad \text{and} \quad g_0(\rho) = \frac{\lambda}{\rho} \ ,
$$

respectively. The boundary condition $g_0(\rho)\to 0$ for $\rho\to 0$ cannot be satisfied for $\lambda\neq 0$; we are thus compelled to set $\lambda=0$. Given that $f(\rho,t)=f_0(\rho)$ and $g(\rho,t)=g_0(\rho)$, the solution of the system (8) is

$$
f(\rho,t) = \frac{\mu_0 I_1 \rho^2}{4A} + C \ , \quad g(\rho,t) \equiv 0 \tag{11}
$$

As is easy to check, by the first of Eqs. (10) it follows that $I_n=0$ for $n>1$. Therefore $I(t)$ is linear in t, i.e., is of the form $I(t)=I_0+I_1 t$. By assuming the initial condition $I(0)=0$, we have that $I_0=0$ and

$$
I(t) = I_1\, t \tag{12}
$$

On the other hand, by integrating Eq. (1): $\sigma'(t)=I(t)/A$, and by assuming that the capacitor is initially uncharged [$\sigma(0)=0$], we get:

$$
\sigma(t) = \frac{I_1 t^2}{2A} \tag{13}
$$

Finally, by Eqs. (5), (11), (12) and (13) the e/m field in the interior of the capacitor is

$$\vec{E} = \left(\frac{I_1 t^2}{2\varepsilon_0 A} + \frac{\mu_0 I_1 \rho^2}{4A} \right) \hat{u}_z \ , \qquad \vec{B} = \frac{\mu_0 I_1 t \rho}{2A} \hat{u}_\varphi \tag{14}$$

where we have set C=0 since, in view of the assumed initial conditions, there is no electric field inside the capacitor if I_1=0. In order for the solution (14) to be valid, the current $I(t)$ charging the capacitor must vary linearly with time, according to (12).

3. Solutions of Maxwell's equations outside the capacitor

We recall that the positive and the negative plate of the capacitor of Fig. 1 are centered at z=0 and z=d, respectively, on the z-axis, where the plate separation d is much smaller than the radius a of the plates. The space exterior to the capacitor consists of points with $\rho > 0$ and $z \notin (0,d)$, as well as points with $\rho > a$ and $0 < z < d$. (In the former case we exclude points on the z-axis, with ρ=0, to ensure the finiteness of our solutions in that region.) We assume that the current $I(t)$ is of "infinite" extent and hence the magnitude of the e/m field is practically z-independent.

The e/m field outside the capacitor is usually described mathematically by the equations [4,5,8]

$$\vec{E} = 0 \ , \quad \vec{B} = \frac{\mu_0 I(t)}{2\pi\rho} \hat{u}_\varphi \tag{15}$$

As the case is with the standard solutions in the interior of the capacitor, the solutions (15) fail to satisfy the Faraday-Henry law (4c) (although they do satisfy the remaining three Maxwell equations), since $\vec{\nabla} \times \vec{E} = 0$ while

$$\frac{\partial \vec{B}}{\partial t} = \frac{\mu_0 I'(t)}{2\pi\rho} \hat{u}_\varphi \ .$$

As before, an exception occurs if the current I is constant in time, i.e., if the capacitor is being charged at a constant rate, so that $I'(t)$=0.

To find more general solutions that satisfy the entire set of the Maxwell equations, we work as in the previous section. Taking into account Lemma 2 in Appendix II, we assume the following general form of the e/m field everywhere outside the capacitor:

$$\vec{E} = f(\rho,t)\, \hat{u}_z \ , \qquad \vec{B} = \left(\frac{\mu_0 I(t)}{2\pi\rho} + g(\rho,t) \right) \hat{u}_\varphi \tag{16}$$

where f and g are functions to be determined consistently with the given current function $I(t)$. The solutions (16) automatically satisfy the first two Maxwell equations (4a) and (4b). By Eqs. (4c) and (4d) we get the following system of PDEs:

$$\begin{aligned} \frac{\partial f}{\partial \rho} &= \frac{\partial g}{\partial t} + \frac{\mu_0 I'(t)}{2\pi\rho} \\ \frac{\partial(\rho g)}{\partial \rho} &= \varepsilon_0 \mu_0 \frac{\partial(\rho f)}{\partial t} \end{aligned} \tag{17}$$

Again, the usual solution with $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$ is possible only if $I'(t)=0$, i.e., if the capacitor is being charged at a constant rate. Note also that, since now $\rho\neq 0$, the boundary condition (7) for g no longer applies.

As we did in the previous section, we seek a series solution of the system (17) in powers of t. We thus expand f, g and I as in Eqs. (9), substitute the expansions into the system (17), and compare terms with equal powers of t. The result is a new recursive system of PDEs:

$$\begin{aligned} f_n'(\rho) &= (n+1)\left[g_{n+1}(\rho) + \frac{\mu_0}{2\pi\rho} I_{n+1} \right] \\ \left[\rho g_n(\rho)\right]' &= (n+1)\,\varepsilon_0 \mu_0 \,\rho f_{n+1}(\rho) \end{aligned} \tag{18}$$

for $n=0,1,2,...$ Again, an obvious solution is the trivial solution $f_n(\rho)\equiv 0$ and $g_n(\rho)\equiv 0$ for all $n=0,1,2,...$, corresponding to $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$. This requires that $I_{n+1}=0$ for all $n=0,1,2,...$, so that $I(t)=I_0=$constant (independent of t).

As in Sec. 2, we seek time-independent solutions for f and g, so that $f=f_0(\rho)$ and $g=g_0(\rho)$ while $f_n(\rho)=0$ and $g_n(\rho)=0$ for $n>0$. The system (18) for $n=0$ gives

$$f_0'(\rho) = \frac{\mu_0 I_1}{2\pi\rho} \quad \text{and} \quad \left[\rho g_0(\rho)\right]' = 0$$

with solutions

$$f_0(\rho) = \frac{\mu_0 I_1}{2\pi} \ln(\kappa\rho) \quad \text{and} \quad g_0(\rho) = \frac{\lambda}{2\pi\rho} \ ,$$

respectively (remember that $\rho>0$), where κ is a positive constant quantity having dimensions of inverse length, and where a factor of 2π has been put in $g_0(\rho)$ for future convenience. Given that $f(\rho,t)=f_0(\rho)$ and $g(\rho,t)=g_0(\rho)$, the solution of the system (17) is

$$f(\rho,t) = \frac{\mu_0 I_1}{2\pi} \ln(\kappa\rho) \ , \quad g(\rho,t) = \frac{\lambda}{2\pi\rho} \tag{19}$$

By the first of Eqs. (18) it follows that $I_n=0$ for $n>1$. Therefore $I(t)$ is linear in t, of the form $I(t)=I_0+I_1 t$. By assuming the initial condition $I(0)=0$, we have that $I_0=0$ and

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$$I(t) = I_1\, t \tag{20}$$

In view of the above results, the e/m field (16) in the exterior of the capacitor is

$$\begin{aligned} \vec{E} &= \frac{\mu_0 I_1}{2\pi} \ln(\kappa\rho)\, \hat{u}_z \ , \\ \vec{B} &= \frac{\mu_0 I_1 t + \lambda}{2\pi\rho}\, \hat{u}_\varphi \end{aligned} \tag{21}$$

For this solution to be valid, the current $I(t)$ must vary linearly with time.

By comparing Eqs. (14) and (21) we observe that the value of the electric field inside the capacitor does not match the value of this field outside for $\rho=a$, where a is the radius of the capacitor. This discontinuity of the electric field at the boundary of the space occupied by the capacitor is a typical characteristic of capacitor problems, in general. On the other hand, in order that the magnetic field in the strip $0 < z < d$ be continuous for $\rho=a$, the expression for $\vec{B}$ in (21) must match the corresponding expression in (14) upon substituting $\rho=a$ and by taking into account that $A=\pi a^2$. This requires that we set $\lambda=0$ in (21), so that this equation finally becomes

$$\begin{aligned} \vec{E} &= \frac{\mu_0 I_1}{2\pi} \ln(\kappa\rho)\, \hat{u}_z \ , \\ \vec{B} &= \frac{\mu_0 I_1 t}{2\pi\rho}\, \hat{u}_\varphi \end{aligned} \tag{22}$$

4. Discussion

As we have seen, expressions for the e/m field inside and outside a charging capacitor may be sought in the general form given by Eqs. (5) and (16), respectively. These expressions contain two unknown functions $f(\rho,t)$ and $g(\rho,t)$ which, in view of Maxwell's equations, satisfy the systems of PDEs (8) and (17). These PDEs, in turn, admit series solutions in powers of t, of the form (9), where it is assumed that the current $I(t)$ itself may be expanded in this fashion.

The coefficients of expansion of f and g may be determined, in principle, by means of the recursion relations (10) and (18), both of which are of the general form

$$\begin{aligned} & f_n'(\rho) = (n+1)\left[g_{n+1}(\rho) + h(\rho) I_{n+1}\right] \\ & \left[\rho g_n(\rho)\right]' = (n+1)\, \varepsilon_0 \mu_0\, \rho f_{n+1}(\rho) \end{aligned} \tag{23}$$

This is not an easy system to integrate, so we are compelled to make certain *ad hoc* assumptions. Suppose, e.g., that we seek a solution such that $f_n(\rho)=0$ and $g_n(\rho)=0$ for $n>k$ ($k\geq 0$). It then follows from the first of Eqs. (23) that $I_{n+1}=0$ for $n>k$ or, equivalently, $I_n=0$ for $n>k+1$. Thus, if $k=0$, $I(t)$ must be linear in t; if $k=1$, $I(t)$ must be quadratic in t; etc.

For a current varying sufficiently slowly with time, we may approximately assume that $I_n=0$ for $n>0$, so that $I(t)=I_0=const$. This allows for the possibility that f and g vanish identically, as is effectively assumed (though not always stated explicitly) in the literature. On the other hand, any smoothly varying $I(t)$ may be assumed to vary linearly with time for a very short time period. Then, a solution of the form (14) and (22) is admissible.

There are several aspects of the solutions described by Eqs. (14) and (22) that may look unphysical: (*a*) the electric field in (22) apparently diverges for $\rho\to\infty$; (*b*) the magnetic field in both (14) and (22) diverges for $t\to\infty$; (*c*) although, by assumption, there are no charges at the interface between the interior and the exterior of the capacitor (i.e., on the cylindrical surface defined by $0<z<d$ and $\rho=a$) the electric field is non-continuous on that surface, contrary to the general boundary conditions required by Maxwell's equations; (*d*) the constant κ in (22) appears to be arbitrary. We may thus use the above solutions only as approximate ones for values of ρ not much larger than the radius a of the plates, as well as for short time intervals. (Note that ρ has to be much smaller than the length of the wire that charges the capacitor if this wire is to be considered of "infinite" length, hence if the external e/m field is to be regarded as z-independent.) We may smoothen the discontinuity problem of the electric field for $\rho=a$ by assuming that this field is continuous at $t=0$, i.e., at the moment when the charging of the capacitor begins. By setting $\rho=a$ in (14) and (22) and by equating the corresponding expressions for $\vec{E}$ we may then determine the value of the constant κ in (22). The result is: $\kappa=e^{1/2}/a$.

For an enlightening discussion of the subtleties concerning the e/m field produced by an infinitely long straight current, the reader is referred to Example 7.9 of [3].

Appendix I. Vector operators in cylindrical coordinates

Let $\vec{A}$ be a vector field, expressed in cylindrical coordinates (ρ, φ, z) as

$$\vec{A} = A_\rho(\rho,\varphi,z)\,\hat{u}_\rho + A_\varphi(\rho,\varphi,z)\,\hat{u}_\varphi + A_z(\rho,\varphi,z)\,\hat{u}_z \ .$$

The *div* and the *rot* of this field in this system of coordinates are written, respectively, as follows:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z} \ ,$$

$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{u}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{u}_\varphi + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial\varphi}\right)\hat{u}_z \ .$$

In particular, if the vector field is of the form

$$\vec{A} = A_\varphi(\rho)\hat{u}_\varphi + A_z(\rho)\hat{u}_z \ ,$$

then $\vec{\nabla}\cdot\vec{A} = 0$.

Appendix II. General form of the electric field

To justify the general expression for the electric field implied in the *Ansatz* (5) used to find solutions of Maxwell's equations inside the capacitor, we need to prove the following:

Lemma 1. If the magnetic field inside the capacitor is azimuthal, of the form

$$\vec{B} = B(\rho,t)\hat{u}_\varphi \tag{A.1}$$

then the electric field (also assumed dependent on ρ and t) is of the form

$$\vec{E} = E(\rho,t)\hat{u}_z \tag{A.2}$$

Proof. Let

$$\vec{E} = E_\rho(\rho,t)\hat{u}_\rho + E_\varphi(\rho,t)\hat{u}_\varphi + E_z(\rho,t)\hat{u}_z \tag{A.3}$$

Then (cf. Appendix I) from Gauss' law (4*a*) it follows that

$$\frac{\partial}{\partial\rho}(\rho E_\rho) = 0 \ \Rightarrow\ E_\rho \equiv \frac{\alpha(t)}{\rho} \tag{A.4}$$

In order for the electric field to be finite at the center of the capacitor (i.e., for ρ=0) we must set $\alpha(t)\equiv 0$, so that $E_\rho(\rho,t)=0$. On the other hand, the *z*-component of Faraday's law (4*c*) yields

$$\frac{\partial}{\partial\rho}(\rho E_\varphi) = 0 \ \Rightarrow\ E_\varphi \equiv \frac{\beta(t)}{\rho} \tag{A.5}$$

Again, finiteness of the electric field for ρ=0 dictates that $\beta(t)\equiv 0$, so that $E_\varphi(\rho,t)=0$. Eventually, only the *z*-component of the electric field is non-vanishing, in accordance with (A.2).

The solutions outside the capacitor are subject to the restriction $\rho>0$. The expression for the electric field implied in the *Ansatz* (16) is based on the following observation:

Lemma 2. If the magnetic field outside the capacitor is azimuthal, of the form (A.1), then the electric field (also assumed dependent on ρ and t) is again of the form (A.2).

Proof. Let the electric field be of the form (A.3). Then from Gauss' law (4*a*) and from the z-component of Faraday's law (4*c*) we get (A.4) and (A.5), respectively. On the other hand, from the ρ- and φ-components of the fourth Maxwell equation (4*d*) we find that $\partial E_\rho/\partial t=0$ and $\partial E_\varphi/\partial t=0$, which means that α and β are actually constants. Thus the general form of the electric field outside the capacitor should be

$$\vec{E}=\frac{\alpha}{\rho}\hat{u}_\rho+\frac{\beta}{\rho}\hat{u}_\varphi+f(\rho,t)\hat{u}_z \ .$$

Obviously, the function $f(\rho,t)$ is related to the time-change of the magnetic field and is expected to vanish if the current I that charges the capacitor is constant. If the electric field itself is to vanish when I=*constant*, both constants α and β must be zero. Eventually, the electric field outside the capacitor must be of the general form (A.2).

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Bäcklund Transformations: Some Old and New Perspectives

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Abstract. Bäcklund transformations (BTs) are traditionally regarded as a tool for integrating nonlinear partial differential equations (PDEs). Their use has been recently extended, however, to problems such as the construction of recursion operators for symmetries of PDEs, as well as the solution of linear systems of PDEs. In this article, the concept and some applications of BTs are reviewed. As an example of an integrable linear system of PDEs, the Maxwell equations of electromagnetism are shown to constitute a BT connecting the wave equations for the electric and the magnetic field; plane-wave solutions of the Maxwell system are constructed in detail. The connection between BTs and recursion operators is also discussed.

Keywords: Bäcklund transformations, integrable systems, Maxwell equations, electromagnetic waves

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1. INTRODUCTION

Bäcklund transformations (BTs) were originally devised as a tool for obtaining solutions of nonlinear partial differential equations (PDEs) (see, e.g., [1] and the references therein). They were later also proven useful as *recursion operators* for constructing infinite sequences of nonlocal symmetries and conservation laws of certain PDEs [2–6].

In simple terms, a BT is a system of PDEs connecting two fields that are required to independently satisfy two respective PDEs [say, (*a*) and (*b*)] in order for the system to be integrable for either field. If a solution of PDE (*a*) is known, then a solution of PDE (*b*) is obtained simply by integrating the BT, without having to actually solve the latter PDE (which, presumably, would be a much harder task). In the case where the PDEs (*a*) and (*b*) are identical, the *auto-BT* produces new solutions of PDE (*a*) from old ones.

As described above, a BT is an auxiliary tool for finding solutions of a given (usually nonlinear) PDE, using known solutions of the same or another PDE. But, what if the BT itself is the differential system whose solutions we are looking for? As it turns out, to solve the problem we need to have parameter-dependent solutions of *both* PDEs (*a*) and (*b*) at hand. By properly matching the parameters (provided this is possible) a solution of the given system is obtained.

The above method is particularly effective in *linear* problems, given that parametric solutions of linear PDEs are generally not hard to find. An important paradigm of a BT associated with a linear problem is offered by the Maxwell system of equations of electromagnetism [7,8]. As is well known, the consistency of this system demands that both the electric and the magnetic field independently satisfy a respective wave equation. These equations have known, parameter-dependent solutions; namely, monochromatic plane waves with arbitrary amplitudes, frequencies and wave vectors (the "parameters" of the problem). By inserting these solutions into the Maxwell system, one may find the appropriate expressions for the "parameters" in order for the plane waves to also be solutions of Maxwell's equations; that is, in order to represent an actual electromagnetic field.

This article, written for educational purposes, is an introduction to the concept of a BT and its application to the solution of PDEs or systems of PDEs. Both "classical" and novel views of a BT are discussed, the former view predominantly concerning integration of nonlinear PDEs while the latter one being applicable mostly to linear systems of PDEs. The article is organized as follows:

In Section 2 we review the classical concept of a BT. The solution-generating process by using a BT is demonstrated in a number of examples.

In Sec. 3 a different perception of a BT is presented, according to which it is the BT itself whose solutions are sought. The concept of *conjugate solutions* is introduced.

As an example, in Secs. 4 and 5 the Maxwell equations in empty space and in a linear conducting medium, respectively, are shown to constitute a BT connecting the wave equations for the electric and the magnetic field. Following [7], the process of constructing plane-wave solutions of this BT is presented in detail. This process is, of course, a familiar problem of electrodynamics but is seen here under a new perspective by employing the concept of a BT.

Finally, in Sec. 6 we briefly review the connection between BTs and recursion operators for generating infinite sequences of nonlocal symmetries of PDEs.

2. BÄCKLUND TRANSFORMATIONS: CLASSICAL VIEWPOINT

Consider two PDEs $P[u]=0$ and $Q[v]=0$ for the unknown functions u and v, respectively. The expressions $P[u]$ and $Q[v]$ may contain the corresponding variables u and v, as well as partial derivatives of u and v with respect to the independent variables. For simplicity, we assume that u and v are functions of only two variables x, t. Partial derivatives with respect to these variables will be denoted by using subscripts: u_x, u_t, u_{xx}, u_{tt}, u_{xt}, etc.

Independently, for the moment, also consider a pair of coupled PDEs for u and v:

$$B_1[u,v]=0 \quad (a) \qquad B_2[u,v]=0 \quad (b) \tag{1}$$

where the expressions $B_i[u,v]$ (i=1,2) may contain u, v as well as partial derivatives of u and v with respect to x and t. We note that u appears in both equations (a) and (b). The question then is: if we find an expression for u by integrating (a) for a given v, will it match the corresponding expression for u found by integrating (b) for the same v? The answer is that, in order that (a) and (b) be consistent with each other for solution for u, the function v must be properly chosen so as to satisfy a certain *consistency condition* (or *integrability condition* or *compatibility condition*).

By a similar reasoning, in order that (a) and (b) in (1) be mutually consistent for solution for v, for some given u, the function u must now itself satisfy a corresponding integrability condition.

If it happens that the two consistency conditions for integrability of the system (1) are precisely the PDEs $P[u]=0$ and $Q[v]=0$, we say that the above system constitutes a *Bäcklund* *transformation* (BT) connecting solutions of $P[u]=0$ with solutions of $Q[v]=0$. In the special case where $P\equiv Q$, i.e., when u and v satisfy *the same* PDE, the system (1) is called an *auto-Bäcklund* transformation (auto-BT) for this PDE.

Suppose now that we seek solutions of the PDE $P[u]=0$. Assume that we are able to find a BT connecting solutions u of this equation with solutions v of the PDE $Q[v]=0$ (if $P\equiv Q$, the auto-BT connects solutions u and v of the same PDE) and let $v=v_0(x,t)$ be some known solution of $Q[v]=0$. The BT is then a system of PDEs for the unknown u,

$$B_i[u, v_0]=0\,, \quad i=1,2 \tag{2}$$

The system (2) is integrable for u, given that the function v_0 satisfies *a priori* the required integrability condition $Q[v]=0$. The solution u then of the system satisfies the PDE $P[u]=0$. Thus a solution $u(x,t)$ of the latter PDE is found without actually solving the equation itself, simply by integrating the BT (2) with respect to u. Of course, this method will be useful provided that integrating the system (2) for u is simpler than integrating the PDE $P[u]=0$ itself. If the transformation (2) is an auto-BT for the PDE $P[u]=0$, then, starting with a known solution $v_0(x,t)$ of this equation and integrating the system (2), we find another solution $u(x,t)$ of the same equation.

Let us see some examples of the use of a BT to generate solutions of a PDE:

1. The *Cauchy-Riemann relations* of Complex Analysis,

$$u_x = v_y \quad (a) \qquad u_y = -v_x \quad (b) \tag{3}$$

(here, the variable t has been renamed y) constitute an auto-BT for the *Laplace equation*,

$$P[w] \equiv w_{xx} + w_{yy} = 0 \tag{4}$$

Let us explain this: Suppose we want to solve the system (3) for u, for a given choice of the function $v(x,y)$. To see if the PDEs (a) and (b) match for solution for u, we must compare them in some way. We thus differentiate (a) with respect to y and (b) with respect to x, and equate the mixed derivatives of u. That is, we apply the integrability condition $(u_x)_y = (u_y)_x$. In this way we eliminate the variable u and find the condition that must be obeyed by $v(x,y)$:

$$P[v] \equiv v_{xx} + v_{yy} = 0 \ .$$

Similarly, by using the integrability condition $(v_x)_y = (v_y)_x$ to eliminate v from the system (3), we find the necessary condition in order that this system be integrable for v, for a given function $u(x,y)$:

$$P[u] \equiv u_{xx} + u_{yy} = 0 \ .$$

In conclusion, the integrability of system (3) with respect to either variable requires that the other variable must satisfy the Laplace equation (4).

Let now $v_0(x,y)$ be a known solution of the Laplace equation (4). Substituting $v=v_0$ in the system (3), we can integrate this system with respect to u. It is not hard to show (by eliminating v_0 from the system) that the solution u will also satisfy the Laplace equation (4). As an example, by choosing the solution $v_0(x,y)=xy$, we find a new solution $u(x,y)=(x^2-y^2)/2+C$.

2. The *Liouville equation* is written

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$$P[u] \equiv u_{xt} - e^{u} = 0 \quad \Leftrightarrow \quad u_{xt} = e^{u} \tag{5}$$

Due to its nonlinearity, this PDE is hard to integrate directly. A solution is thus sought by means of a BT. We consider an auxiliary function $v(x,t)$ and an associated PDE,

$$Q[v] \equiv v_{xt} = 0 \tag{6}$$

We also consider the system of first-order PDEs,

$$u_x + v_x = \sqrt{2}\; e^{(u-v)/2} \quad (a) \qquad u_t - v_t = \sqrt{2}\; e^{(u+v)/2} \quad (b) \tag{7}$$

Differentiating the PDE (a) with respect to t and the PDE (b) with respect to x, and eliminating ($u_t - v_t$) and ($u_x + v_x$) in the ensuing equations with the aid of (a) and (b), we find that u and v satisfy the PDEs (5) and (6), respectively. Thus, the system (7) is a BT connecting solutions of (5) and (6). Starting with the trivial solution v=0 of (6), and integrating the system

$$u_x = \sqrt{2}\; e^{u/2} \;, \qquad u_t = \sqrt{2}\; e^{u/2} \;,$$

we find a nontrivial solution of (5):

$$u(x,t) = -2\ln\left(C - \frac{x+t}{\sqrt{2}} \right) .$$

3. The “*sine-Gordon” equation* has applications in various areas of Physics, e.g., in the study of crystalline solids, in the transmission of elastic waves, in magnetism, in elementary-particle models, etc. The equation (whose name is a pun on the related linear Klein-Gordon equation) is written

$$P[u] \equiv u_{xt} - \sin u = 0 \quad \Leftrightarrow \quad u_{xt} = \sin u \tag{8}$$

The following system of equations is an auto-BT for the nonlinear PDE (8):

$$\frac{1}{2}(u+v)_x = a\,\sin\left(\frac{u-v}{2}\right) \;, \qquad \frac{1}{2}(u-v)_t = \frac{1}{a}\,\sin\left(\frac{u+v}{2}\right) \tag{9}$$

where a (≠0) is an arbitrary real constant. [Because of the presence of a, the system (9) is called a *parametric* BT.] When u is a solution of (8) the BT (9) is integrable for v, which, in turn, also is a solution of (8): $P[v]$=0; and vice versa. Starting with the trivial solution v=0 of v_{xt}= sin v , and integrating the system

$$u_x = 2a\,\sin\frac{u}{2} \;, \qquad u_t = \frac{2}{a}\,\sin\frac{u}{2} \;,$$

we obtain a new solution of (8):

$$u(x,t) = 4\arctan\left\{ C\exp\left(ax + \frac{t}{a} \right) \right\} .$$

3. CONJUGATE SOLUTIONS AND ANOTHER VIEW OF A BT

As presented in the previous section, a BT is an auxiliary device for constructing solutions of a (usually nonlinear) PDE from known solutions of the same or another PDE. The converse problem, where solutions of the differential system representing the BT itself are sought, is also of interest, however, and has been recently suggested [7,8] in connection with the Maxwell equations (see subsequent sections).

To be specific, assume that we need to integrate a given system of PDEs connecting two functions *u* and *v*:

$$B_i[u,v]=0 , \quad i=1,2 \tag{10}$$

Suppose that the integrability of the system for both functions requires that *u* and *v* separately satisfy the respective PDEs

$$P[u]=0 \quad (a) \qquad Q[v]=0 \quad (b) \tag{11}$$

That is, the system (10) is a BT connecting solutions of the PDEs (11). Assume, now, that these PDEs possess known (or, in any case, easy to find) *parameter-dependent solutions* of the form

$$u=f(x,y;\alpha,\beta,\ldots) \quad , \qquad v=g(x,y;\kappa,\lambda,\ldots) \tag{12}$$

where *α*, *β*, *κ*, *λ*, etc., are (real or complex) parameters. If values of these parameters can be determined for which *u* and *v* jointly satisfy the system (10), we say that the solutions *u* and *v* of the PDEs (11*a*) and (11*b*), respectively, are *conjugate through the BT* (10) (or *BT-conjugate*, for short). By finding a pair of BT-conjugate solutions one thus automatically obtains a solution of the system (10).

Note that solutions of *both* integrability conditions *P*[*u*]=0 and *Q*[*v*]=0 must now be known in advance! From the practical point of view the method is thus most applicable in *linear* problems, since it is much easier to find parameter-dependent solutions of the PDEs (11) in this case.

Let us see an example: Going back to the Cauchy-Riemann relations (3), we try the following parametric solutions of the Laplace equation (4):

$$u(x,y)=\alpha(x^2-y^2)+\beta x+\gamma y \ ,$$
$$v(x,y)=\kappa xy+\lambda x+\mu y \ .$$

Substituting these into the BT (3), we find that *κ*=2*α*, *μ*=*β* and *λ*= –*γ*. Therefore, the solutions

$$u(x,y)=\alpha(x^2-y^2)+\beta x+\gamma y \ ,$$
$$v(x,y)=2\alpha xy-\gamma x+\beta y$$

of the Laplace equation are BT-conjugate through the Cauchy-Riemann relations.

As a counter-example, let us try a different combination:

$$u(x,y)=\alpha xy \ , \quad v(x,y)=\beta xy \ .$$

Inserting these into the system (3) and taking into account the independence of *x* and *y*, we find that the only possible values of the parameters *α* and *β* are *α*=*β*=0, so that *u*(*x*,*y*)= *v*(*x*,*y*)=0. Thus, no non-trivial BT-conjugate solutions exist in this case.

4. EXAMPLE: THE MAXWELL EQUATIONS IN EMPTY SPACE

An example of an integrable linear system whose solutions are of physical interest is furnished by the *Maxwell equations* of electrodynamics. Interestingly, as noted recently [7], the Maxwell system has the property of a BT whose integrability conditions are the electromagnetic (e/m) wave equations that are separately valid for the electric and the magnetic field. These equations possess parameter-dependent solutions that, by a proper choice of the parameters, can be made BT-conjugate through the Maxwell system. In this and the following section we discuss the BT property of the Maxwell equations in vacuum and in a conducting medium, respectively.

In *empty space*, where no charges or currents (whether free or bound) exist, the Maxwell equations are written (in S.I. units) [9]

$$\begin{aligned} &(a) \quad \vec{\nabla}\cdot\vec{E}=0 \qquad &&(c) \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b) \quad \vec{\nabla}\cdot\vec{B}=0 \qquad &&(d) \quad \vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{13}$$

where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. Here we have a system of four PDEs for two fields. The question is: what are the necessary conditions that each of these fields must satisfy in order for the system (13) to be self-consistent? In other words, what are the *consistency conditions* (or *integrability conditions*) for this system?

Guided by our experience from Sec. 2, to find these conditions we perform various differentiations of the equations of system (13) and require that certain differential identities be satisfied. Our aim is, of course, to eliminate one field (electric or magnetic) in favor of the other and find some higher-order PDE that the latter field must obey.

As can be checked, two differential identities are satisfied automatically in the system (13):

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E})=0 \ , \quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})=0 \ ,$$

$$(\vec{\nabla}\cdot\vec{E})_t=\vec{\nabla}\cdot\vec{E}_t \ , \quad (\vec{\nabla}\cdot\vec{B})_t=\vec{\nabla}\cdot\vec{B}_t \ .$$

Two others read

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \tag{14}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \tag{15}$$

Taking the *rot* of (13*c*) and using (14), (13*a*) and (13*d*), we find

$$\nabla^2 \vec{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{16}$$

Similarly, taking the *rot* of (13*d*) and using (15), (13*b*) and (13*c*), we get

$$\nabla^2 \vec{B} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \tag{17}$$

No new information is furnished by the remaining two integrability conditions,

$$(\vec{\nabla} \times \vec{E})_t = \vec{\nabla} \times \vec{E}_t \quad , \quad (\vec{\nabla} \times \vec{B})_t = \vec{\nabla} \times \vec{B}_t \quad .$$

Note that we have *uncoupled* the equations for the two fields in the system (13), deriving separate second-order PDEs for each field. Putting

$$\varepsilon_0 \mu_0 \equiv \frac{1}{c^2} \quad \Leftrightarrow \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \tag{18}$$

(where *c* is the speed of light in vacuum) we rewrite (16) and (17) in wave-equation form:

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{19}$$

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \tag{20}$$

We conclude that the Maxwell system (13) is a BT relating solutions of the e/m wave equations (19) and (20), these equations representing the integrability conditions of the BT. It should be noted that this BT is *not* an *auto*-BT! Indeed, although the PDEs (19) and (20) are of similar form, they concern *different* fields with different physical dimensions and physical properties.

The e/m wave equations admit plane-wave solutions of the form $\vec{F}(\vec{k} \cdot \vec{r} - \omega t)$, with

$$\frac{\omega}{k} = c \quad \text{where} \quad k = |\vec{k}| \tag{21}$$

The simplest such solutions are *monochromatic plane waves* of angular frequency *ω*, propagating in the direction of the wave vector $\vec{k}$:

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0 \exp\{i(\vec{k} \cdot \vec{r} - \omega t)\} \qquad (a) \\ \vec{B}(\vec{r},t) &= \vec{B}_0 \exp\{i(\vec{k} \cdot \vec{r} - \omega t)\} \qquad (b) \end{aligned} \tag{22}$$

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where $\vec{E}_0$ and $\vec{B}_0$ are constant complex amplitudes. The constants appearing in the above equations (amplitudes, frequency and wave vector) can be chosen arbitrarily; thus they can be regarded as *parameters* on which the plane waves (22) depend.

We must note carefully that, although every pair of fields $(\vec{E},\vec{B})$ satisfying the Maxwell equations (13) also satisfies the wave equations (19) and (20), the converse is not true. Thus, the plane-wave solutions (22) are not *a priori* solutions of the Maxwell system (i.e., do not represent actual e/m fields). This problem can be taken care of, however, by a proper choice of the parameters in (22). To this end, we substitute the general solutions (22) into the BT (13) to find the extra conditions the latter system demands. By fixing the wave parameters, the two wave solutions in (22) will become *BT-conjugate* through the Maxwell system (13).

Substituting (22*a*) and (22*b*) into (13*a*) and (13*b*), respectively, and taking into account that $\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}$, we have

$$(\vec{E}_0\, e^{-i\omega t})\cdot\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = 0 \;\Rightarrow\; (\vec{k}\cdot\vec{E}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0\,,$$
$$(\vec{B}_0\, e^{-i\omega t})\cdot\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = 0 \;\Rightarrow\; (\vec{k}\cdot\vec{B}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0\,,$$

so that

$$\vec{k}\cdot\vec{E}_0 = 0\,,\quad \vec{k}\cdot\vec{B}_0 = 0\,. \tag{23}$$

Relations (23) reflect the fact that that the monochromatic plane e/m wave is a *transverse wave*.

Next, substituting (22*a*) and (22*b*) into (13*c*) and (13*d*), we find

$$e^{-i\omega t}(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}})\times\vec{E}_0 = i\,\omega\,\vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow$$
$$(\vec{k}\times\vec{E}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \omega\,\vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,,$$

$$e^{-i\omega t}(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}})\times\vec{B}_0 = -i\,\omega\,\varepsilon_0\mu_0\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow$$
$$(\vec{k}\times\vec{B}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = -\frac{\omega}{c^2}\,\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,,$$

so that

$$\vec{k}\times\vec{E}_0 = \omega\,\vec{B}_0\,,\quad \vec{k}\times\vec{B}_0 = -\frac{\omega}{c^2}\,\vec{E}_0 \tag{24}$$

We note that the fields $\vec{E}$ and $\vec{B}$ are normal to each other, as well as normal to the direction of propagation of the wave. We also remark that the two vector equations in (24) are not independent of each other, since, by cross-multiplying the first relation by $\vec{k}$, we get the second relation.

Introducing a unit vector $\hat{\tau}$ in the direction of the wave vector $\vec{k}$,

$$\hat{\tau} = \vec{k}/k \quad (k = |\vec{k}| = \omega/c)\,,$$

we rewrite the first of equations (24) as

$$\vec{B}_0 = \frac{k}{\omega}(\hat{\tau} \times \vec{E}_0) = \frac{1}{c}(\hat{\tau} \times \vec{E}_0) \ .$$

The BT-conjugate solutions in (22) are now written

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0 \exp\{i(\vec{k} \cdot \vec{r} - \omega t)\} \ , \\ \vec{B}(\vec{r},t) &= \frac{1}{c}(\hat{\tau} \times \vec{E}_0) \exp\{i(\vec{k} \cdot \vec{r} - \omega t)\} = \frac{1}{c}\,\hat{\tau} \times \vec{E} \end{aligned} \tag{25}$$

As constructed, the complex vector fields in (25) satisfy the Maxwell system (13). Since this system is homogeneous linear with real coefficients, the real parts of the fields (25) also satisfy it. To find the expressions for the real solutions (which, after all, carry the physics of the situation) we take the simplest case of *linear polarization* and write

$$\vec{E}_0 = \vec{E}_{0,R}\, e^{i\alpha} \tag{26}$$

where the vector $\vec{E}_{0,R}$ as well as the number α are real. The *real* versions of the fields (25), then, read

$$\begin{aligned} \vec{E} &= \vec{E}_{0,R} \cos(\vec{k} \cdot \vec{r} - \omega t + \alpha) \ , \\ \vec{B} &= \frac{1}{c}(\hat{\tau} \times \vec{E}_{0,R}) \cos(\vec{k} \cdot \vec{r} - \omega t + \alpha) = \frac{1}{c}\,\hat{\tau} \times \vec{E} \end{aligned} \tag{27}$$

We note, in particular, that the fields $\vec{E}$ and $\vec{B}$ “oscillate” in phase.

Our results for the Maxwell equations in vacuum can be extended to the case of a *linear non-conducting medium* upon replacement of ε_0 and μ_0 with ε and μ, respectively. The speed of propagation of the e/m wave is, in this case,

$$\upsilon = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}} \ .$$

In the next section we study the more complex case of a linear medium having a finite conductivity.

5. EXAMPLE: THE MAXWELL SYSTEM FOR A LINEAR CONDUCTING MEDIUM

Consider a linear conducting medium of conductivity σ. In such a medium, Ohm’s law is satisfied: $\vec{J}_f = \sigma\vec{E}$, where $\vec{J}_f$ is the free current density. The Maxwell equations take on the form [9]

$$\begin{aligned}
&(a) \quad \vec{\nabla}\cdot\vec{E}=0 \qquad && (c) \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\
&(b) \quad \vec{\nabla}\cdot\vec{B}=0 \qquad && (d) \quad \vec{\nabla}\times\vec{B}=\mu\sigma\vec{E}+\varepsilon\mu\frac{\partial\vec{E}}{\partial t}
\end{aligned} \tag{28}$$

By requiring satisfaction of the integrability conditions

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}\ ,$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B}\ ,$$

we obtain the *modified wave equations*

$$\begin{aligned}
&\nabla^2\vec{E}-\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}-\mu\sigma\frac{\partial\vec{E}}{\partial t}=0 \\
&\nabla^2\vec{B}-\varepsilon\mu\frac{\partial^2\vec{B}}{\partial t^2}-\mu\sigma\frac{\partial\vec{B}}{\partial t}=0
\end{aligned} \tag{29}$$

which must be separately satisfied by each field. As in Sec. 4, no further information is furnished by the remaining integrability conditions.

The linear differential system (28) is a BT relating solutions of the wave equations (29). As in the vacuum case, this BT is *not* an auto-BT. We now seek BT-conjugate solutions. As can be verified by direct substitution into equations (29), these PDEs admit parameter-dependent solutions of the form

$$\begin{aligned}
\vec{E}(\vec{r},t)&=\vec{E}_0\exp\{-s\,\hat{\tau}\cdot\vec{r}+i\,(\vec{k}\cdot\vec{r}-\omega t)\} \\
&=\vec{E}_0\exp\left\{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}\right\}\exp(-i\omega t)\ , \\
\vec{B}(\vec{r},t)&=\vec{B}_0\exp\{-s\,\hat{\tau}\cdot\vec{r}+i\,(\vec{k}\cdot\vec{r}-\omega t)\} \\
&=\vec{B}_0\exp\left\{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}\right\}\exp(-i\omega t)
\end{aligned} \tag{30}$$

where $\hat{\tau}$ is the unit vector in the direction of the wave vector $\vec{k}$:

$$\hat{\tau}=\vec{k}/k \quad (k=|\vec{k}|=\omega/\upsilon)$$

(υ is the speed of propagation of the wave inside the conducting medium) and where, for given physical characteristics ε, μ, σ of the medium, the parameters s, k and ω satisfy the algebraic system

$$s^2-k^2+\varepsilon\mu\omega^2=0\ , \quad \mu\sigma\omega-2sk=0 \tag{31}$$

We note that, for arbitrary choices of the amplitudes $\vec{E}_0$ and $\vec{B}_0$, the vector fields (30) are not *a priori* solutions of the Maxwell system (28), thus are not BT-conjugate solutions. To obtain such solutions we substitute expressions (30) into the system (28). With the aid of the relation

$$\vec{\nabla} e^{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}} = \left(i-\frac{s}{k}\right)\vec{k}\, e^{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}}$$

one can show that (28*a*) and (28*b*) impose the conditions

$$\vec{k}\cdot\vec{E}_0 = 0\,, \quad \vec{k}\cdot\vec{B}_0 = 0 \tag{32}$$

As in the vacuum case, the e/m wave in a conducting medium is a *transverse* wave.

By substituting (30) into (28*c*) and (28*d*), two more conditions are found:

$$(k+is)\,\hat{\tau}\times\vec{E}_0 = \omega\vec{B}_0 \tag{33}$$

$$(k+is)\,\hat{\tau}\times\vec{B}_0 = -(\varepsilon\mu\omega + i\mu\sigma)\vec{E}_0 \tag{34}$$

Note, however, that (34) is not an independent equation since it can be reproduced by cross-multiplying (33) by $\hat{\tau}$, taking into account the algebraic relations (31).

The BT-conjugate solutions of the wave equations (29) are now written

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0\, e^{-s\,\hat{\tau}\cdot\vec{r}}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,, \\ \vec{B}(\vec{r},t) &= \frac{k+is}{\omega}\,(\hat{\tau}\times\vec{E}_0)\, e^{-s\,\hat{\tau}\cdot\vec{r}}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \end{aligned} \tag{35}$$

To find the corresponding real solutions, we assume linear polarization of the wave, as before, and set

$$\vec{E}_0 = \vec{E}_{0,R}\, e^{i\alpha}\,.$$

We also put

$$k + i\,s = |\,k + i\,s\,|e^{i\varphi} = \sqrt{k^2+s^2}\; e^{i\varphi} \;\;;\;\; \tan\varphi = s/k\,.$$

Taking the real parts of equations (35), we finally have:

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_{0,R}\, e^{-s\,\hat{\tau}\cdot\vec{r}} \cos(\vec{k}\cdot\vec{r} - \omega t + \alpha)\,, \\ \vec{B}(\vec{r},t) &= \frac{\sqrt{k^2+s^2}}{\omega}\,(\hat{\tau}\times\vec{E}_{0,R})\, e^{-s\,\hat{\tau}\cdot\vec{r}} \cos(\vec{k}\cdot\vec{r} - \omega t + \alpha + \varphi)\,. \end{aligned}$$

As an exercise, the student may show that these results reduce to those for a linear non-conducting medium (cf. Sec. 4) in the limit $\sigma \to 0$.

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6. BTS AS RECURSION OPERATORS

The concept of symmetries of PDEs was discussed in [1]. Let us review the main facts:
Consider a PDE $F[u]=0$, where, for simplicity, $u=u(x,t)$. A transformation

$$u(x,t) \rightarrow u'(x,t)$$

from the function u to a new function u' represents a *symmetry* of the given PDE if the following condition is satisfied: $u'(x,t)$ is a solution of $F[u]=0$ *if* $u(x,t)$ is a solution. That is,

$$F[u']=0 \quad when \quad F[u]=0 \tag{36}$$

An *infinitesimal symmetry transformation* is written

$$u' = u+\delta u = u+\alpha Q[u] \tag{37}$$

where α is an infinitesimal parameter. The function $Q[u]\equiv Q(x, t, u, u_x , u_t ,...)$ is called the *symmetry characteristic* of the transformation (37).

In order that a function $Q[u]$ be a symmetry characteristic for the PDE $F[u]=0$, it must satisfy a certain PDE that expresses the *symmetry condition* for $F[u]=0$. We write, symbolically,

$$S(Q;u)=0 \quad when \quad F[u]=0 \tag{38}$$

where the expression S depends *linearly* on Q and its partial derivatives. Thus, (38) is a linear PDE for Q, in which equation the variable u enters as a sort of parametric function that is required to satisfy the PDE $F[u]=0$.

A *recursion operator* $\hat{R}$ [10] is a linear operator which, acting on a symmetry characteristic Q, produces a new symmetry characteristic $Q' = \hat{R}Q$. That is,

$$S(\hat{R}Q;u)=0 \quad when \quad S(Q;u)=0 \tag{39}$$

It is not too difficult to show that *any power of a recursion operator also is a recursion operator*. This means that, starting with any symmetry characteristic Q, one may in principle obtain an infinite set of characteristics (thus, an infinite number of symmetries) by repeated application of the recursion operator.

A new approach to recursion operators was suggested in the early 1990s [2,3] (see also [4-6]). According to this view, a recursion operator is an auto-BT for the linear PDE (38) expressing the symmetry condition of the problem; that is, a BT producing new solutions Q' of (38) from old ones, Q. Typically, this type of BT produces *nonlocal* symmetries, i.e., symmetry characteristics depending on *integrals* (rather than derivatives) of u.

As an example, consider the *chiral field equation*

$$F[g] \equiv (g^{-1}g_x)_x + (g^{-1}g_t)_t = 0 \tag{40}$$

(as usual, subscripts denote partial differentiations) where g is a $GL(n,C)$-valued function of x and t (i.e., an invertible complex $n\times n$ matrix, differentiable for all x, t).

Let $Q[g]$ be a symmetry characteristic of the PDE (40). It is convenient to put

$$Q[g] = g\,\Phi[g]$$

and write the corresponding infinitesimal symmetry transformation in the form

$$g' = g + \delta g = g + \alpha g \Phi[g] \tag{41}$$

The symmetry condition that Q must satisfy will be a PDE linear in Q, thus in Φ also. As can be shown [4], this PDE is

$$S(\Phi; g) \equiv \Phi_{xx} + \Phi_{tt} + [g^{-1}g_x, \Phi_x] + [g^{-1}g_t, \Phi_t] = 0 \tag{42}$$

which must be valid when $F[g]=0$ (where, in general, $[A, B] \equiv AB-BA$ denotes the *commutator* of two matrices A and B).

For a given g satisfying $F[g]=0$, consider now the following system of PDEs for the matrix functions Φ and Φ':

$$\begin{aligned} \Phi'_x &= \Phi_t + [g^{-1}g_t\,, \Phi] \\ -\Phi'_t &= \Phi_x + [g^{-1}g_x\,, \Phi] \end{aligned} \tag{43}$$

The integrability condition $(\Phi'_x)_t = (\Phi'_t)_x$, together with the equation $F[g]=0$, require that Φ be a solution of (42): $S(\Phi; g) = 0$. Similarly, by the integrability condition $(\Phi_t)_x = (\Phi_x)_t$ one finds, after a lengthy calculation: $S(\Phi'; g) = 0$.

In conclusion, for any g satisfying the PDE (40), the system (43) is a BT relating solutions Φ and Φ' of the symmetry condition (42) of this PDE; that is, relating different symmetries of the chiral field equation (40). Thus, if a symmetry characteristic $Q=g\Phi$ of (40) is known, a new characteristic $Q'=g\Phi'$ may be found by integrating the BT (43); the converse is also true. Since the BT (43) produces new symmetries from old ones, it may be regarded as a *recursion operator* for the PDE (40).

As an example, for any constant matrix M the choice $\Phi=M$ clearly satisfies the symmetry condition (42). This corresponds to the symmetry characteristic $Q=gM$. By integrating the BT (43) for Φ', we get $\Phi'=[X, M]$ and $Q'=g[X, M]$, where X is the "potential" of the PDE (40), defined by the system of PDEs

$$X_x = g^{-1}g_t \;, \quad -X_t = g^{-1}g_x \tag{44}$$

Note the *nonlocal* character of the BT-produced symmetry Q', due to the presence of the potential X. Indeed, as seen from (44), in order to find X one has to *integrate* the chiral field g with respect to the independent variables x and t. The above process can be continued indefinitely by repeated application of the recursion operator (43), leading to an infinite sequence of increasingly nonlocal symmetries.

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7. SUMMARY

Classically, Bäcklund transformations (BTs) have been developed as a useful tool for finding solutions of nonlinear PDEs, given that these equations are usually hard to solve by direct methods. By means of examples we saw that, starting with even the most trivial solution of a PDE, one may produce a highly nontrivial solution of this (or another) PDE by integrating the BT, without solving the original, nonlinear PDE directly (which, in most cases, is a much harder task).

A different use of BTs, that was recently proposed [7,8], concerns predominantly the solution of linear systems of PDEs. This method relies on the existence of parameter-dependent solutions of the linear PDEs expressing the integrability conditions of the BT. This time it is the BT itself (rather than its associated integrability conditions) whose solutions are sought.

An appropriate example for demonstrating this approach to the concept of a BT is furnished by the Maxwell equations of electromagnetism. We showed that this system of PDEs can be treated as a BT whose integrability conditions are the wave equations for the electric and the magnetic field. These wave equations have known, parameter-dependent solutions – monochromatic plane waves – with arbitrary amplitudes, frequencies and wave vectors playing the roles of the “parameters”. By substituting these solutions into the BT, one may determine the required relations among the parameters in order that these plane waves also represent electromagnetic fields (i.e., in order that they be solutions of the Maxwell system). The results arrived at by this method are, of course, well known in advanced electrodynamics. The process of deriving them, however, is seen here in a new light by employing the concept of a BT.

BTs have also proven useful as *recursion operators* for deriving infinite sets of nonlocal symmetries and conservation laws of PDEs [2-6] (see also [11] and the references therein). Specifically, the BT produces an increasingly nonlocal sequence of symmetry characteristics, i.e., solutions of the linear equation expressing the symmetry condition (or “linearization”) of a given PDE.

An interesting conclusion is that the concept of a BT, which has been proven useful for integrating nonlinear PDEs, may also have important applications in linear problems. Research on these matters is in progress.

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The Maxwell equations as a Bäcklund transformation

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Abstract

Bäcklund transformations (BTs) are a useful tool for integrating nonlinear partial differential equations (PDEs). However, the significance of BTs in linear problems should not be ignored. In fact, an important linear system of PDEs in Physics, namely, the Maxwell equations of electromagnetism, may be viewed as a BT relating the wave equations for the electric and the magnetic field, these equations representing integrability conditions for solution of the Maxwell system. We examine the BT property of this system in detail, both for the vacuum case and for the case of a linear conducting medium.

1. Introduction

Bäcklund transformations (BTs) are an effective tool for integrating partial differential equations (PDEs). They are particularly useful for obtaining solutions of nonlinear PDEs, given that these equations are often notoriously hard to solve by direct methods (see [1] and the references therein).

Generally speaking, given two PDEs – say (*a*) and (*b*) – for the unknown functions *u* and *v*, respectively, a BT relating these PDEs is a system of auxiliary PDEs containing both *u* and *v*, such that the consistency (*integrability*) of this system requires that the original PDEs (*a*) and (*b*) be separately satisfied. Then, if a solution of PDE (*a*) is known, a solution of PDE (*b*) is found simply by integrating the BT, without having to integrate the PDE (*b*) directly (which, presumably, is a much harder task).

In addition to being a solution-generating mechanism, BTs may also serve as *recursion operators* for obtaining infinite hierarchies of (generally nonlocal) symmetries and conservation laws of a PDE [1–7]. It is by this method that the full symmetry Lie algebra of the self-dual Yang-Mills equation was found [3,6].

In this article, the nature of which is mostly pedagogical, we adopt a somewhat different (in a sense, inverse) view of a BT, suitable for the treatment of linear problems. Suppose we are given a system of PDEs for the unknown functions *u* and *v*. Suppose, further, that the consistency of this system requires that two PDEs, one for *u* and one for *v*, be separately satisfied (thus, the given system is a BT connecting these PDEs). The PDEs are assumed to possess known solutions for *u* and *v*, each solution depending on a number of parameters. If, by a proper choice of the parameters, these functions are made to satisfy the original differential system, then a solution to this system has been found. In other words, we are seeking solutions of the given system by using known, parameter-dependent solutions of the individual PDEs expressing the integrability conditions of this system. Pairs of functions (*u*,*v*) satisfying the system will be said to represent *BT-conjugate solutions*.

This modified view of the concept of a BT has an important application in electromagnetism that serves as a paradigm for the significance of BTs in linear problems. As discussed in this paper, the Maxwell equations for a linear medium exactly fit this BT scheme. Indeed, as is well known, the consistency of the Maxwell system requires that the electric and the magnetic field satisfy separate wave equations. These equations have known, parameter-dependent solutions, namely, monochromatic plane waves with arbitrary amplitudes, wave vectors, frequencies, etc. (the "parameters" of the problem). By inserting these solutions into the Maxwell system, one may find the necessary conditions on the parameters in order that the plane waves for the two fields represent BT-conjugate solutions of Maxwell's equations.

The paper is organized as follows:

Section 2 reviews the classical concept of a BT. The solution-generating process by using a BT is demonstrated in a number of examples.

In Sec. 3 the concept of parametric, BT-conjugate solutions is introduced. A simple example illustrates the idea.

In Sec. 4 the Maxwell equations in empty space are shown to constitute a BT in the sense described in Sec. 3. For completeness of presentation (and for the benefit of the student) the process of constructing BT-conjugate plane-wave solutions is presented in detail.

Finally, in Sec. 5 the Maxwell system for a linear conducting medium is similarly examined.

The results of Secs. 4 and 5 are, of course, well known from classical electromagnetic theory. It is mathematically interesting, however, to revisit the problem of constructing solutions of Maxwell's equations from a novel point of view by using the concept of a BT and by treating the electric and the magnetic component of a plane e/m wave as BT-conjugate solutions.

2. Bäcklund transformations: definition and examples

The general idea of a Bäcklund transformation (BT) was explained in [1] (see also the references therein). Let us review the main points:

We consider two PDEs $P[u]=0$ and $Q[v]=0$, where the expressions $P[u]$ and $Q[v]$ may contain the unknown functions u and v, respectively, as well as some of their partial derivatives with respect to the independent variables. For simplicity, we assume that u and v are functions of only two variables x, t. Partial derivatives with respect to these variables will be denoted by using subscripts, e.g., u_x, u_t, u_{xx}, u_{tt}, u_{xt}, etc.

We also consider a system of coupled PDEs for u and v,

$$B_i[u,v]=0\ ,\quad i=1,2 \tag{1}$$

where the expressions $B_i[u,v]$ may contain u, v and certain of their partial derivatives with respect to x and t. The system (1) is assumed to be integrable for v (the two equations are compatible with each other for solution for v) when u satisfies the PDE $P[u]=0$. The solution v, then, satisfies the PDE $Q[v]=0$. Conversely, the system (1) is integrable for u if v satisfies the PDE $Q[v]=0$, the solution u then satisfying $P[u]=0$.

If the above assumptions are valid, we say that the system (1) constitutes a BT connecting solutions of $P[u]=0$ with solutions of $Q[v]=0$. In the special case where $P\equiv Q$, i.e., when u and v satisfy the same PDE, the system (1) is called an *auto-Bäcklund* transformation (auto-BT).

Suppose now that we seek solutions of the PDE $P[u]=0$. Also, assume that we possess a BT connecting solutions u of this equation with solutions v of the PDE $Q[v]=0$ (if $P\equiv Q$ the auto-BT connects solutions u and v of the same PDE). Let $v=v_0(x,t)$ be a known solution of $Q[v]=0$. The BT is then a system of equations for the unknown u:

$$B_i[u,v_0]=0\ ,\quad i=1,2 \tag{2}$$

Given that $Q[v_0]=0$, the system (2) is integrable for u and its solution satisfies the PDE $P[u]=0$. We may thus find a solution $u(x,t)$ of $P[u]=0$ without solving the equation itself, simply by integrating the BT (2) with respect to u. Of course, the use of this method is meaningful provided that we know a solution $v_0(x,t)$ of $Q[v]=0$ beforehand, as well as that integrating the system (2) for u is simpler than integrating the PDE $P[u]=0$ directly. If the transformation (2) is an auto-BT, then, starting with a known solution $v_0(x,t)$ of $P[u]=0$ and integrating the system (2), we find another solution $u(x,t)$ of the same equation.

Let us see some examples of using a BT to generate solutions of a PDE:

1. The *Cauchy-Riemann relations* of complex analysis,

$$u_x=v_y \quad (a) \qquad u_y=-v_x \quad (b) \tag{3}$$

(here, the variable t has been renamed y) constitute an auto-BT for the (linear) *Laplace equation*,

$$P[w]\equiv w_{xx}+w_{yy}=0 \tag{4}$$

Indeed, differentiating (3a) with respect to y and (3b) with respect to x, and demanding that the *integrability condition* $(u_x)_y=(u_y)_x$ be satisfied, we eliminate the variable u to find the consistency condition that must be obeyed by $v(x,y)$ in order that the system (3) be integrable for u:

$$P[v]\equiv v_{xx}+v_{yy}=0\ .$$

Conversely, eliminating v from the system (3) by using the integrability condition $(v_x)_y=(v_y)_x$, we find the necessary condition for u in order for the system to be integrable for v:

$$P[u]\equiv u_{xx}+u_{yy}=0\ .$$

Now, let $v_0(x,y)$ be a known solution of the Laplace equation (4). Substituting $v=v_0$ in the system (3), we can integrate the latter with respect to u to find another solution of the Laplace equation. For example, by choosing $v_0(x,y)=xy$ we find the solution $u(x,y)=(x^2-y^2)/2+C$.

2. The *Liouville equation* is written

$$P[u]\equiv u_{xt}-e^u=0 \quad \Leftrightarrow \quad u_{xt}=e^u \tag{5}$$

Solving the PDE (5) directly is a difficult task in view of this equation's nonlinearity. A solution can be found, however, by using a BT. We thus consider an auxiliary function $v(x,t)$ and an associated linear PDE,

$$Q[v]\equiv v_{xt}=0 \tag{6}$$

We also consider the system of first-order PDEs,

$$\begin{aligned} u_x+v_x&=\sqrt{2}\ e^{(u-v)/2}\\ u_t-v_t&=\sqrt{2}\ e^{(u+v)/2} \end{aligned} \tag{7}$$

It can be shown that the self-consistency of the system (7) requires that u and v independently satisfy the PDEs (5) and (6), respectively. Thus, this system constitutes a BT connecting solutions of (5) and (6). Starting with the trivial solution $v=0$ of (6) and integrating the system

$$u_x=\sqrt{2}\ e^{u/2}\ ,\qquad u_t=\sqrt{2}\ e^{u/2}\ ,$$

we find a solution of (5):

$$u(x,t)=-2\ln\left(C-\frac{x+t}{\sqrt{2}}\right)\ .$$

3. The *"sine-Gordon" equation* has applications in various areas of Physics, such as in the study of crystalline solids, in the transmission of elastic waves, in magnetism, in elementary-particle models, etc. The equation (whose name

is a pun on the related linear Klein-Gordon equation) is written

$$u_{xt} = \sin u \qquad (8)$$

As can be proven, the differential system

$$\begin{aligned} \frac{1}{2}(u+v)_x &= a \sin\left(\frac{u-v}{2}\right) \\ \frac{1}{2}(u-v)_t &= \frac{1}{a} \sin\left(\frac{u+v}{2}\right) \end{aligned} \qquad (9)$$

[where a ($\neq 0$) is an arbitrary real constant] is a parametric auto-BT for the PDE (8). Starting with the trivial solution $v=0$ of $v_{xt} = \sin v$, and integrating the system

$$u_x = 2a \sin\frac{u}{2} \quad , \quad u_t = \frac{2}{a} \sin\frac{u}{2} \quad ,$$

we obtain a new solution of (8):

$$u(x,t) = 4 \arctan\left\{C \exp\left(ax + \frac{t}{a}\right)\right\} \ .$$

3. BT-conjugate solutions

Consider a system of coupled PDEs for the functions u and v of two independent variables x, y:

$$B_i[u,v] = 0 \ , \quad i = 1,2 \qquad (10)$$

Assume that the integrability of this system for both u and v requires that the following PDEs be independently satisfied:

$$P[u] = 0 \quad (a) \qquad Q[v] = 0 \quad (b) \qquad (11)$$

That is, the system (10) represents a BT connecting the PDEs (11). Assume, further, that the PDEs (11) possess parameter-dependent solutions of the form

$$\begin{aligned} u &= f(x, y; \alpha, \beta, \gamma, \ldots) \ , \\ v &= g(x, y; \kappa, \lambda, \mu, \ldots) \end{aligned} \qquad (12)$$

where α, β, κ, λ, etc., are (real or complex) parameters. If values of these parameters can be determined for which u and v satisfy the system (10), we say that the solutions u and v of the PDEs (11a) and (11b), respectively, are *conjugate through the BT* (10) (or *BT-conjugate*, for short).

Let us see an example: Going back to the Cauchy-Riemann relations (3), we try the following parametric solutions of the Laplace equation (4):

$$\begin{aligned} u(x,y) &= \alpha(x^2 - y^2) + \beta x + \gamma y \ , \\ v(x,y) &= \kappa xy + \lambda x + \mu y \ . \end{aligned}$$

Substituting these into the BT (3), we find that $\kappa=2\alpha$, $\mu=\beta$ and $\lambda=-\gamma$. Therefore, the solutions

$$\begin{aligned} u(x,y) &= \alpha(x^2 - y^2) + \beta x + \gamma y \ , \\ v(x,y) &= 2\alpha xy - \gamma x + \beta y \end{aligned}$$

of the Laplace equation are BT-conjugate through the Cauchy-Riemann relations.

As a counter-example, let us try a different combination:

$$u(x,y) = \alpha xy \quad , \quad v(x,y) = \beta xy \ .$$

Inserting these into the system (3) and taking into account the independence of x and y, we find that the only possible values of the parameters α and β are $\alpha=\beta=0$, so that $u(x,y)= v(x,y)=0$. Thus, no non-trivial BT-conjugate solutions exist in this case.

4. Application to the Maxwell equations in empty space

As is well known, according to the Maxwell theory all electromagnetic (e/m) disturbances propagate in space as waves running at the speed of light. It is interesting from the mathematical point of view that the vacuum wave equations for the electric and the magnetic field are connected to each other through the Maxwell system of equations in much the same way two PDEs are connected via a Bäcklund transformation. In fact, certain parameter-dependent solutions of the two wave equations are BT-conjugate through the Maxwell system.

In empty space, where no charges or currents (whether free or bound) exist, the Maxwell equations are written in S.I. units [8]:

$$\begin{aligned} &(a)\ \vec{\nabla}\cdot\vec{E} = 0 \qquad (c)\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ &(b)\ \vec{\nabla}\cdot\vec{B} = 0 \qquad (d)\ \vec{\nabla}\times\vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \qquad (13)$$

where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. In order that this system of PDEs be self-consistent (thus integrable for the two fields), certain consistency conditions (or *integrability conditions*) must be satisfied. Four are satisfied automatically:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = 0 \quad , \quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 \quad ,$$

$$(\vec{\nabla}\cdot\vec{E})_t = \vec{\nabla}\cdot\vec{E}_t \quad , \quad (\vec{\nabla}\cdot\vec{B})_t = \vec{\nabla}\cdot\vec{B}_t \ .$$

Two others read:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \tag{14}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \tag{15}$$

Taking the *rot* of (13*c*) and using (14), (13*a*) and (13*d*), we find:

$$\nabla^2\vec{E}-\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}=0 \tag{16}$$

Similarly, taking the *rot* of (13*d*) and using (15), (13*b*) and (13*c*), we get:

$$\nabla^2\vec{B}-\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}=0 \tag{17}$$

No new information is furnished by the remaining two integrability conditions,

$$(\vec{\nabla}\times\vec{E})_t=\vec{\nabla}\times\vec{E}_t \quad , \quad (\vec{\nabla}\times\vec{B})_t=\vec{\nabla}\times\vec{B}_t \quad .$$

Putting

$$\varepsilon_0\mu_0\equiv\frac{1}{c^2} \quad\Leftrightarrow\quad c=\frac{1}{\sqrt{\varepsilon_0\mu_0}} \tag{18}$$

we rewrite Eqs. (16) and (17) in wave-equation form:

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0 \tag{19}$$

$$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}=0 \tag{20}$$

The PDEs (19) and (20) are consistency conditions that must be separately satisfied by $\vec{E}$ and $\vec{B}$ in order that the differential system (13) be integrable for either field, given the value of the other field. In other words, the system (13) is a BT relating solutions of the wave equations (19) and (20).

It should be noted carefully that the BT (13) is *not* an *auto*-BT! Indeed, although the PDEs (19) and (20) look similar, they concern *different* fields with different physical dimensions and physical properties. A true auto-BT should connect similar objects (such as, e.g., different mathematical expressions for the electric field).

The above wave equations admit plane-wave solutions of the form $\vec{F}(\vec{k}\cdot\vec{r}-\omega t)$, with

$$\frac{\omega}{k}=c \quad \text{where} \quad k=|\vec{k}| \tag{21}$$

The simplest such solutions are *monochromatic plane waves* of angular frequency *ω*, propagating in the direction of the wave vector $\vec{k}$:

$$\begin{aligned}\vec{E}(\vec{r},t)&=\vec{E}_0\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \qquad (a)\\ \vec{B}(\vec{r},t)&=\vec{B}_0\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \qquad (b)\end{aligned} \tag{22}$$

where the $\vec{E}_0$ and $\vec{B}_0$ represent constant complex amplitudes. Since all constants appearing in equations (22) (that is, amplitudes, frequency and wave vector) can be arbitrarily chosen, they can be regarded as *parameters* on which the solutions (22) of the wave equations depend.

Clearly, although every pair of fields $(\vec{E},\vec{B})$ that satisfies the Maxwell equations (13) also satisfies the respective wave equations (19) and (20), the converse is not true. This means that the solutions (22) of the wave equation are not *a priori* solutions of the Maxwell system of equations (i.e., do not represent e/m fields). This problem can be remedied, however, by appropriate choice of the parameters. To this end, we substitute the general solutions (22) into the system (13) in order to find the extra conditions this system requires; that is, in order to make the two functions in (22) BT-conjugate solutions of the respective wave equations (19) and (20).

Substituting (22*a*) and (22*b*) into (13*a*) and (13*b*), respectively, and taking into account that $\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=i\vec{k}\,e^{i\vec{k}\cdot\vec{r}}$, we have:

$$(\vec{E}_0e^{-i\omega t})\cdot\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=0 \;\Rightarrow\; (\vec{k}\cdot\vec{E}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=0\;,$$
$$(\vec{B}_0e^{-i\omega t})\cdot\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=0 \;\Rightarrow\; (\vec{k}\cdot\vec{B}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=0\;,$$

so that

$$\vec{k}\cdot\vec{E}_0=0\;,\quad \vec{k}\cdot\vec{B}_0=0\,. \tag{23}$$

Physically, this means that the monochromatic plane e/m wave is a *transverse* wave.

Next, substituting (22*a*) and (22*b*) into (13*c*) and (13*d*), we find:

$$e^{-i\omega t}(\vec{\nabla}e^{i\vec{k}\cdot\vec{r}})\times\vec{E}_0=i\omega\,\vec{B}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} \;\Rightarrow$$
$$(\vec{k}\times\vec{E}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=\omega\,\vec{B}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\;,$$

$$e^{-i\omega t}(\vec{\nabla}e^{i\vec{k}\cdot\vec{r}})\times\vec{B}_0=-i\omega\,\varepsilon_0\mu_0\vec{E}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} \;\Rightarrow$$
$$(\vec{k}\times\vec{B}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=-\frac{\omega}{c^2}\,\vec{E}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\;,$$

so that

$$\vec{k}\times\vec{E}_0=\omega\,\vec{B}_0\ ,\qquad \vec{k}\times\vec{B}_0=-\frac{\omega}{c^2}\,\vec{E}_0 \tag{24}$$

This means that the fields $\vec{E}$ and $\vec{B}$ are normal to each other as well as being normal to the direction of propagation. It can be seen that the two vector equations in (24) are not independent of each other; indeed, cross-multiplying the first relation by $\vec{k}$ we get the second one.

Introducing a unit vector $\hat{\tau}$ in the direction of the wave vector $\vec{k}$,

$$\hat{\tau}=\vec{k}/k\qquad (k=|\vec{k}|=\omega/c)\ ,$$

we rewrite the first of Eqs. (24) as

$$\vec{B}_0=\frac{k}{\omega}(\hat{\tau}\times\vec{E}_0)=\frac{1}{c}(\hat{\tau}\times\vec{E}_0)\ .$$

The BT-conjugate solutions in (22) are now written:

$$\begin{aligned}\vec{E}(\vec{r},t)&=\vec{E}_0\exp\{\,i(\vec{k}\cdot\vec{r}-\omega t)\}\ ,\\ \vec{B}(\vec{r},t)&=\frac{1}{c}(\hat{\tau}\times\vec{E}_0)\exp\{\,i(\vec{k}\cdot\vec{r}-\omega t)\}\\ &=\frac{1}{c}\,\hat{\tau}\times\vec{E}\end{aligned} \tag{25}$$

As constructed, the complex vector fields in (25) satisfy the Maxwell system (13), which is a homogeneous linear system with real coefficients. Evidently, the real parts of these fields also satisfy this system. To find the expressions for the real solutions (which, after all, carry the physics of the situation) we take the simplest case of a linearly polarized e/m wave and write:

$$\vec{E}_0=\vec{E}_{0,R}\,e^{\,i\alpha} \tag{26}$$

where the vector $\vec{E}_{0,R}$ and the number α are real. The *real* versions of the fields (25), then, read:

$$\begin{aligned}\vec{E}&=\vec{E}_{0,R}\cos(\vec{k}\cdot\vec{r}\,-\,\omega\,t+\alpha)\ ,\\ \vec{B}&=\frac{1}{c}(\hat{\tau}\times\vec{E}_{0,R})\cos(\vec{k}\cdot\vec{r}\,-\,\omega\,t+\alpha)\\ &=\frac{1}{c}\,\hat{\tau}\times\vec{E}\end{aligned} \tag{27}$$

We note, in particular, that the fields $\vec{E}$ and $\vec{B}$ "oscillate" in phase.

Our results for the Maxwell equations in vacuum can be extended to the case of a *linear non-conducting medium* upon replacement of ε_0 and μ_0 with ε and μ, respectively. The speed of propagation of the e/m wave is, in this case,

$$\upsilon=\frac{\omega}{k}=\frac{1}{\sqrt{\varepsilon\mu}}\ .$$

5. The Maxwell system for a linear conducting medium

In a linear conducting medium of conductivity σ, in which Ohm's law is satisfied, $\vec{J}_f=\sigma\vec{E}$ (where $\vec{J}_f$ is the free current density), the Maxwell equations read [8]:

$$\begin{aligned}&(a)\ \ \vec{\nabla}\cdot\vec{E}=0 \qquad (c)\ \ \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\\ &(b)\ \ \vec{\nabla}\cdot\vec{B}=0 \qquad (d)\ \ \vec{\nabla}\times\vec{B}=\mu\sigma\vec{E}+\varepsilon\mu\,\frac{\partial\vec{E}}{\partial t}\end{aligned} \tag{28}$$

By the integrability conditions

$$\begin{aligned}&\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}\ ,\\ &\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B}\ ,\end{aligned}$$

we get the *modified wave equations*

$$\begin{aligned}&\nabla^2\vec{E}\,-\,\varepsilon\mu\,\frac{\partial^2\vec{E}}{\partial t^2}\,-\,\mu\sigma\,\frac{\partial\vec{E}}{\partial t}=0\\ &\nabla^2\vec{B}\,-\,\varepsilon\mu\,\frac{\partial^2\vec{B}}{\partial t^2}\,-\,\mu\sigma\,\frac{\partial\vec{B}}{\partial t}=0\end{aligned} \tag{29}$$

No new information is furnished by the remaining integrability conditions (cf. Sec. 4).

We observe that the linear differential system (28) is a BT relating solutions of the wave equations (29) (as explained in the previous section, this BT is *not* an auto-BT). As in the vacuum case, we seek BT-conjugate such solutions. As can be verified by direct substitution into Eqs. (29), these PDEs admit parametric plane-wave solutions of the form

$$\begin{aligned}\vec{E}(\vec{r},t)&=\vec{E}_0\exp\{-s\,\hat{\tau}\cdot\vec{r}+i\,(\vec{k}\cdot\vec{r}-\omega t)\}\\ &=\vec{E}_0\exp\left\{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}\right\}\exp(-i\omega t)\ ,\\ \vec{B}(\vec{r},t)&=\vec{B}_0\exp\{-s\,\hat{\tau}\cdot\vec{r}+i\,(\vec{k}\cdot\vec{r}-\omega t)\}\\ &=\vec{B}_0\exp\left\{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}\right\}\exp(-i\omega t)\end{aligned} \tag{30}$$

where $\hat{\tau}$ is the unit vector in the direction of the wave vector $\vec{k}$,

$$\hat{\tau} = \vec{k} / k \quad (k = |\vec{k}| = \omega / \upsilon)$$

(υ is the speed of propagation of the wave inside the conducting medium) and where, for given physical characteristics ε, μ, σ of the medium, the parameters s, k and ω satisfy the algebraic system

$$\begin{aligned} & s^2 - k^2 + \varepsilon \mu \omega^2 = 0 \ , \\ & \mu \sigma \omega - 2sk = 0 \end{aligned} \tag{31}$$

Up to this point the complex amplitudes $\vec{E}_0$ and $\vec{B}_0$ in relations (30) are arbitrary and the vector fields (30) are not *a priori* solutions of the Maxwell equations (28), thus are not yet BT-conjugate solutions of the respective wave equations in (29). To find the restrictions these amplitudes must satisfy, we insert Eqs. (30) into the system (28). With the aid of the relation

$$\vec{\nabla} e^{\left(i - \frac{s}{k}\right) \vec{k} \cdot \vec{r}} = \left(i - \frac{s}{k} \right) \vec{k} \, e^{\left(i - \frac{s}{k}\right) \vec{k} \cdot \vec{r}} \ ,$$

it is not hard to show that (28*a*) and (28*b*) impose the conditions

$$\vec{k} \cdot \vec{E}_0 = 0 \ , \quad \vec{k} \cdot \vec{B}_0 = 0 \tag{32}$$

Again, this means that the e/m wave is a transverse wave.

Substituting (30) into (28*c*) and (28*d*), we find two more conditions:

$$(k + is) \, \hat{\tau} \times \vec{E}_0 = \omega \vec{B}_0 \tag{33}$$

$$(k + is) \, \hat{\tau} \times \vec{B}_0 = -(\varepsilon \mu \omega + i \mu \sigma) \vec{E}_0 \tag{34}$$

However, (34) is not an independent equation since it can be reproduced by cross-multiplication of (33) by $\hat{\tau}$ and use of relations (31).

The BT-conjugate solutions of the wave equations (29) are now written:

$$\begin{aligned} & \vec{E}(\vec{r}, t) = \vec{E}_0 \, e^{-s \hat{\tau} \cdot \vec{r}} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \ , \\ & \vec{B}(\vec{r}, t) = \frac{k + is}{\omega} \, (\hat{\tau} \times \vec{E}_0) \, e^{-s \hat{\tau} \cdot \vec{r}} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \end{aligned} \tag{35}$$

To find the corresponding real solutions, we assume linear polarization of the e/m wave and set, as before,

$$\vec{E}_0 = \vec{E}_{0,R} \, e^{i\alpha} \ .$$

We also set

$$\begin{aligned} & k + i s = |k + i s| e^{i\varphi} = \sqrt{k^2 + s^2} \ e^{i\varphi} \ ; \\ & \tan \varphi = s / k \ . \end{aligned}$$

Taking the real parts of Eqs. (35), we finally have:

$$\begin{aligned} & \vec{E}(\vec{r}, t) = \vec{E}_{0,R} \, e^{-s \hat{\tau} \cdot \vec{r}} \cos(\vec{k} \cdot \vec{r} - \omega t + \alpha) \ , \\ & \vec{B}(\vec{r}, t) = \frac{\sqrt{k^2 + s^2}}{\omega} \, (\hat{\tau} \times \vec{E}_{0,R}) \, e^{-s \hat{\tau} \cdot \vec{r}} \cos(\vec{k} \cdot \vec{r} - \omega t + \alpha + \varphi) \ . \end{aligned}$$

6. Summary and concluding remarks

Bäcklund transformations (BTs) were originally devised as a tool for finding solutions of nonlinear partial differential equations (PDEs). They were later also proven useful as nonlocal recursion operators for constructing infinite sequences of symmetries and conservation laws of certain PDEs [2–7].

Generally speaking, a BT is a system of PDEs connecting two fields that are required to independently satisfy two respective PDEs in order for the system to be integrable for either field. If a solution of either PDE is known, then a solution of the other PDE is obtained by integrating the BT, without having to actually solve the latter PDE explicitly (which, presumably, would be a much harder task). In the case where the two PDEs are identical, an auto-BT produces new solutions of a PDE from old ones.

As described above, a BT is an auxiliary tool for finding solutions of a given (usually nonlinear) PDE, using known solutions of the same or another PDE. In this article, however, we approached the BT concept differently by actually inverting the problem. According to this scheme, it is the solutions of the BT itself that we are after, having parameter-dependent solutions of the PDEs that express the integrability conditions at hand. By a proper choice of the parameters, a pair of solutions of these PDEs may possibly be found that satisfies the given BT. These solutions are then said to be *conjugate* with respect to the BT.

A pedagogical paradigm for demonstrating this particular approach to the concept of a BT is offered by the Maxwell system of equations of electromagnetism. We showed that this system can be thought of as a BT whose integrability conditions are the wave equations for the electric and the magnetic field. These wave equations have known, parameter-dependent solutions (monochromatic plane waves) with arbitrary amplitudes, frequencies, wave vectors, etc. By substituting these solutions into the BT, one may determine the required relations among the parameters in order that the plane waves also represent electromagnetic fields, i.e., are BT-conjugate solutions of the Maxwell system. The results arrived at by this method are, of course, well known in advanced electrodynamics. The process of deriving them, however, is seen here in a new light by employing the concept of a BT.

We remark that the physical situation was examined from the point of view of a fixed inertial observer. Thus, since no spacetime transformations were involved, we used the classical form of the Maxwell equations (with $\vec{E}$ and $\vec{B}$ retaining their individual characters) rather than the manifestly covariant form of these equations.

An interesting conclusion is that the concept of a Bäcklund transformation, which has been proven extremely useful for finding solutions of nonlinear PDEs, can in certain cases also prove useful for integrating *linear systems* of PDEs. Such systems appear often in Physics and Electrical Engineering (see, e.g., [9]) and it would certainly be of interest to explore the possibility of using BT methods for their integration.

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Plane-wave solutions of Maxwell's equations: An educational note

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Synopsis

In electrodynamics courses and textbooks, the properties of plane electromagnetic waves in both conducting and non-conducting media are typically studied from the point of view of the prototype case of a monochromatic plane wave. In this note an approach is suggested that starts from more general considerations and better exploits the independence of the Maxwell equations.

Plane-wave solutions of Maxwell's equations: An educational note

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Abstract

In electrodynamics courses and textbooks, the properties of plane electromagnetic waves in both conducting and non-conducting media are typically studied from the point of view of the prototype case of a monochromatic plane wave. In this note an approach is suggested that starts from more general considerations and better exploits the independence of the Maxwell equations.

1. Introduction

Plane electromagnetic (e/m) waves constitute a significant type of solution of the time-dependent Maxwell equations. A standard educational approach in courses and textbooks (at both the intermediate [1-4] and the advanced [5,6] level; see also [7,8]) is to examine the prototype case of a monochromatic plane wave in both a conducting and a non-conducting medium.

In this note a more general approach to the problem is described that makes minimal initial assumptions regarding the specific functional forms of the plane waves representing the electric and the magnetic field. The only assumption one does need to make from the outset is that both fields (electric and magnetic) are expressible in integral form as linear superpositions of monochromatic waves. In particular, it is not even necessary to *a priori* require that the plane waves representing the two fields travel in the same direction.

In Section 2 we review the case of a monochromatic plane e/m wave in empty space. A more general (non-monochromatic) treatment of the plane-wave propagation problem in empty space is then described in Sec. 3. In Sec. 4 this general approach is extended to plane-wave solutions in the case of a conducting medium; an interesting difference from the monochromatic case is noted.

2. The monochromatic-wave description for empty space

In empty space, where no charges or currents (whether free or bound) exist, the Maxwell equations are written (in S.I. units)

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E}=0 &\qquad &(c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B}=0 &\qquad &(d)\quad \vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \qquad (1)$$

where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. By applying the identities

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \ ,$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \, ,$$

we obtain separate wave equations for $\vec{E}$ and $\vec{B}$:

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0 \qquad (2)$$

$$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}=0 \qquad (3)$$

where

$$c=\frac{1}{\sqrt{\varepsilon_0\mu_0}} \qquad (4)$$

We try monochromatic plane-wave solutions of (2) and (3), of angular frequency *ω*, propagating in the direction of the wave vector $\vec{k}$:

$$\begin{aligned} \vec{E}(\vec{r},t)&=\vec{E}_0\exp\{i\,(\vec{k}\cdot\vec{r}-\omega t)\} \qquad (a)\\ \vec{B}(\vec{r},t)&=\vec{B}_0\exp\{i\,(\vec{k}\cdot\vec{r}-\omega t)\} \qquad (b)\end{aligned} \qquad (5)$$

where $\vec{E}_0$ and $\vec{B}_0$ are constant complex amplitudes, and where

$$\frac{\omega}{k}=c \qquad (k=|\vec{k}\,|) \qquad (6)$$

The general solutions (5) do not *a priori* represent an e/m field. To find the extra constraints required, we must substitute Eqs. (5) into the Maxwell system (1). By taking into account that $\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=i\vec{k}\,e^{i\vec{k}\cdot\vec{r}}$, the *div* equations (1*a*) and (1*b*) yield

$$\vec{k}\cdot\vec{E}=0 \quad (a) \qquad \vec{k}\cdot\vec{B}=0 \quad (b) \qquad (7)$$

while the *rot* equations (1*c*) and (1*d*) give

$$\vec{k}\times\vec{E}=\omega\,\vec{B} \quad (a) \qquad \vec{k}\times\vec{B}=-\frac{\omega}{c^2}\,\vec{E} \quad (b) \qquad (8)$$

Now, we notice that the four equations (7)–(8) do not form an independent set since (7*b*) and (8*b*) can be reproduced by using (7*a*) and (8*a*). Indeed, taking the dot product of (8*a*) with $\vec{k}$ we get (7*b*), while taking the cross product of (8*a*) with $\vec{k}$, and using (7*a*) and (6), we find (8*b*).

So, from 4 independent Maxwell equations we obtained only 2 independent pieces of information. This happened because we “fed” our trial solutions (5) with more information than necessary, in anticipation of results that follow *a posteriori* from Maxwell’s equations. Thus, we assumed from the outset that the two waves (electric and magnetic) have similar simple functional forms and propagate in the same direction. By relaxing these initial assumptions, our analysis acquires a richer and much more interesting structure.

3. A more general approach for empty space

Let us assume, more generally, that the fields $\vec{E}$ and $\vec{B}$ represent plane waves propagating in empty space in the directions of the unit vectors $\hat{\tau}$ and $\hat{\sigma}$, respectively:

$$\vec{E}(\vec{r},t)=\vec{F}(\hat{\tau}\cdot\vec{r}-ct)\ ,\quad \vec{B}(\vec{r},t)=\vec{G}(\hat{\sigma}\cdot\vec{r}-ct) \tag{9}$$

Furthermore, assume that the functions $\vec{F}$ and $\vec{G}$ can be expressed as linear combinations of monochromatic plane waves of the form (5), for continuously varying values of k and ω, where $\omega=ck$, according to (6). Then $\vec{E}$ and $\vec{B}$ can be written in Fourier-integral form, as follows:

$$\begin{aligned}\vec{E}&=\int \vec{E}_0(k)\,e^{ik(\hat{\tau}\cdot\vec{r}-ct)}dk\\ \vec{B}&=\int \vec{B}_0(k)\,e^{ik(\hat{\sigma}\cdot\vec{r}-ct)}dk\end{aligned} \tag{10}$$

In general, the integration variable k is assumed to run from 0 to $+\infty$. For notational economy, the limits of integration with respect to k will not be displayed explicitly.

By setting

$$u=\hat{\tau}\cdot\vec{r}-ct\ ,\quad v=\hat{\sigma}\cdot\vec{r}-ct \tag{11}$$

we write

$$\begin{aligned}\vec{E}(u)&=\int \vec{E}_0(k)\,e^{iku}dk\\ \vec{B}(v)&=\int \vec{B}_0(k)\,e^{ikv}dk\end{aligned} \tag{12}$$

We note that

$$\vec{\nabla}e^{iku}=ik\hat{\tau}e^{iku}\ ,\quad \vec{\nabla}e^{ikv}=ik\hat{\sigma}e^{ikv} \tag{13}$$

By using (12) and (13) we find that

$$\vec{\nabla}\cdot\vec{E}=\int ik\hat{\tau}\cdot\vec{E}_0(k)\,e^{iku}dk\ ,\quad \vec{\nabla}\cdot\vec{B}=\int ik\hat{\sigma}\cdot\vec{B}_0(k)\,e^{ikv}dk\ ,$$

$$\vec{\nabla}\times\vec{E}=\int ik\hat{\tau}\times\vec{E}_0(k)\,e^{iku}dk\ ,\quad \vec{\nabla}\times\vec{B}=\int ik\hat{\sigma}\times\vec{B}_0(k)\,e^{ikv}dk\ .$$

Moreover, we have that

$$\frac{\partial\vec{E}}{\partial t}=-\int i\omega\vec{E}_0(k)\,e^{iku}dk\ ,\quad \frac{\partial\vec{B}}{\partial t}=-\int i\omega\vec{B}_0(k)\,e^{ikv}dk$$

where, as always, $\omega=ck$.

The two Gauss' laws (1*a*) and (1*b*) yield

$$\int k\hat{\tau}\cdot\vec{E}_0(k)\,e^{iku}dk=0 \quad \text{and} \quad \int k\hat{\sigma}\cdot\vec{B}_0(k)\,e^{ikv}dk=0\,,$$

respectively. In order that these relations be valid identically for all *u* and all *v*, respectively, we must have

$$\hat{\tau}\cdot\vec{E}_0(k)=0 \quad \text{and} \quad \hat{\sigma}\cdot\vec{B}_0(k)=0\,, \quad \text{for all } k \tag{14}$$

From Faraday's law (1*c*) and the Ampère-Maxwell law (1*d*) we obtain two more integral equations:

$$\int k\hat{\tau}\times\vec{E}_0(k)\,e^{iku}dk=\int \omega\vec{B}_0(k)\,e^{ikv}dk \tag{15}$$

$$\int k\hat{\sigma}\times\vec{B}_0(k)\,e^{ikv}dk=-\int \frac{\omega}{c^2}\vec{E}_0(k)\,e^{iku}dk \tag{16}$$

where we have taken into account Eq. (4).

Taking the cross product of (15) with $\hat{\sigma}$ and using (16), we find the integral relation

$$\int k[(\hat{\sigma}\cdot\vec{E}_0)\,\hat{\tau}-(\hat{\sigma}\cdot\hat{\tau})\vec{E}_0]\,e^{iku}dk=-\int k\vec{E}_0\,e^{iku}dk\,.$$

This is true for all *u* if

$$(\hat{\sigma}\cdot\vec{E}_0)\,\hat{\tau}-(\hat{\sigma}\cdot\hat{\tau})\vec{E}_0=-\vec{E}_0 \;\Rightarrow\; (\hat{\sigma}\cdot\hat{\tau}-1)\vec{E}_0=(\hat{\sigma}\cdot\vec{E}_0)\,\hat{\tau}\,.$$

Given that, by (14), $\vec{E}_0$ and $\hat{\tau}$ are mutually perpendicular, the above relation can only be valid if $\hat{\sigma}\cdot\hat{\tau}=1$ and $\hat{\sigma}\cdot\vec{E}_0=0$. This, in turn, can only be satisfied if $\hat{\sigma}=\hat{\tau}$. The same conclusion is reached by taking the cross product of (16) with $\hat{\tau}$ and by using (15) as well as the fact that $\vec{B}_0$ is normal to $\hat{\sigma}$. From (11) we then have that

$$u=v=\hat{\tau}\cdot\vec{r}-ct$$

so that relations (12) become

$$\begin{aligned}\vec{E}(\vec{r},t)&=\int\vec{E}_0(k)\,e^{iku}dk=\int\vec{E}_0(k)\,e^{ik(\hat{\tau}\cdot\vec{r}-ct)}dk\\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)\,e^{iku}dk=\int\vec{B}_0(k)\,e^{ik(\hat{\tau}\cdot\vec{r}-ct)}dk\end{aligned} \tag{17}$$

Equations (14) are now rewritten as

$$\hat{\tau}\cdot\vec{E}_0(k)=0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B}_0(k)=0\,, \quad \text{for all } k \tag{18}$$

Furthermore, in order that (15) and (16) (with *u* and $\hat{\tau}$ in place of *v* and $\hat{\sigma}$, respectively) be identically valid for all *u*, we must have

$$k\,\hat{\tau}\times\vec{E}_0(k)=\omega\vec{B}_0(k)\ \Leftrightarrow\ \hat{\tau}\times\vec{E}_0(k)=c\vec{B}_0(k) \tag{19}$$

and

$$k\,\hat{\tau}\times\vec{B}_0(k)=-\frac{\omega}{c^2}\vec{E}_0(k)\ \Leftrightarrow\ \hat{\tau}\times\vec{B}_0(k)=-\frac{1}{c}\vec{E}_0(k) \tag{20}$$

for all k, where $k=\omega/c$. Notice, however, that (19) and (20) are not independent equations, since (20) is essentially the cross product of (19) with $\hat{\tau}$.

In summary, the general plane-wave solutions to the Maxwell system (1) are given by relations (17) with the additional constraints (18) and (19). This is, of course, a well-known result, derived here by starting with more general assumptions and by best exploiting the independence [9] of the Maxwell equations.

Let us summarize our main findings:

1. The fields $\vec{E}$ and $\vec{B}$ are plane waves traveling in the same direction, defined by the unit vector $\hat{\tau}$; these fields satisfy the Maxwell equations in empty space.

2. The e/m wave $(\vec{E},\vec{B})$ is a *transverse* wave. Indeed, from equations (17) and the orthogonality relations (18) it follows that

$$\hat{\tau}\cdot\vec{E}=0 \quad\text{and}\quad \hat{\tau}\cdot\vec{B}=0 \tag{21}$$

3. The fields $\vec{E}$ and $\vec{B}$ are mutually perpendicular. Moreover, the $(\vec{E},\vec{B},\hat{\tau})$ define a right-handed rectangular system. Indeed, by cross-multiplying (17) with $\hat{\tau}$ and by using (19) and (20), we find:

$$\hat{\tau}\times\vec{E}=c\vec{B}\ ,\quad \hat{\tau}\times\vec{B}=-\frac{1}{c}\vec{E} \tag{22}$$

4. Taking *real values* of (21) and (22), we have:

$$\hat{\tau}\cdot\mathrm{Re}\vec{E}=0\ ,\quad \hat{\tau}\cdot\mathrm{Re}\vec{B}=0 \quad\text{and}\quad \hat{\tau}\times\mathrm{Re}\vec{E}=c\,\mathrm{Re}\vec{B} \tag{23}$$

The magnitude of the last vector equation in (23) gives a relation between the instantaneous values of the electric and the magnetic field:

$$|\mathrm{Re}\vec{E}|=c\,|\mathrm{Re}\vec{B}| \tag{24}$$

The above results for empty space can be extended in a straightforward way to the case of a *linear, non-conducting, non-dispersive* medium upon replacement of ε_0 and μ_0 with ε and μ, respectively [3]. The (frequency-independent) speed of propagation of the plane e/m wave in this case is $\upsilon=1/(\varepsilon\mu)^{1/2}$.

4. The case of a conducting medium

The Maxwell equations for a conducting medium of conductivity σ may be written as follows [1,3]:

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E}=0 \qquad &&(c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B}=0 \qquad &&(d)\quad \vec{\nabla}\times\vec{B}=\mu\sigma\vec{E}+\varepsilon\mu\,\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{25}$$

By using the vector identities

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}\ ,$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B}\,,$$

the relations (25) lead to the *modified wave equations*

$$\nabla^2\vec{E}\,-\,\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}\,-\,\mu\sigma\frac{\partial\vec{E}}{\partial t}=0 \tag{26}$$

$$\nabla^2\vec{B}\,-\,\varepsilon\mu\frac{\partial^2\vec{B}}{\partial t^2}\,-\,\mu\sigma\frac{\partial\vec{B}}{\partial t}=0 \tag{27}$$

Guided by our monochromatic-wave approach to the problem in [7,8], we now try a more general, integral form of solution of the above wave equations:

$$\begin{aligned} \vec{E}(\vec{r},t)&=\int\vec{E}_0(k)\,e^{-s\hat{\tau}\cdot\vec{r}}e^{i(k\,\hat{\tau}\cdot\vec{r}-\omega t)}dk=\int\vec{E}_0(k)\exp\left\{(ik-s)\,\hat{\tau}\cdot\vec{r}-i\omega t\right\}dk \\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)\,e^{-s\hat{\tau}\cdot\vec{r}}e^{i(k\,\hat{\tau}\cdot\vec{r}-\omega t)}dk=\int\vec{B}_0(k)\exp\left\{(ik-s)\,\hat{\tau}\cdot\vec{r}-i\omega t\right\}dk \end{aligned} \tag{28}$$

where s is a real parameter related to the conductivity of the medium. As in the vacuum case, the unit vector $\hat{\tau}$ indicates the direction of propagation of the wave. Notice that we have assumed from the outset that both waves – electric and magnetic – propagate in the same direction, in view of the fact that our results must agree with those for a non-conducting medium (in particular, for the vacuum) upon setting $s=0$.

It is convenient to set

$$\exp\left\{(ik-s)\,\hat{\tau}\cdot\vec{r}-i\,\omega t\right\}\equiv A(\vec{r},t) \tag{29}$$

Then, Eq. (28) takes on the form

$$\begin{aligned} \vec{E}(\vec{r},t)&=\int\vec{E}_0(k)\,A(\vec{r},t)\,dk \\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)\,A(\vec{r},t)\,dk \end{aligned} \tag{30}$$

The following relations can be easily proven:

$$\vec{\nabla}A(\vec{r},t)=(i\,k-s)\,\hat{\tau}A(\vec{r},t) \tag{31}$$

$$\nabla^2A(\vec{r},t)=(s^2-k^2-2isk)A(\vec{r},t) \tag{32}$$

Moreover,

$$\frac{\partial}{\partial t}A(\vec{r},t)=-i\omega A(\vec{r},t) \quad \text{and} \quad \frac{\partial^2}{\partial t^2}A(\vec{r},t)=-\omega^2 A(\vec{r},t)\ .$$

From (26) we get

$$\int[(s^2-k^2+\varepsilon\mu\omega^2)+i(\mu\sigma\omega-2sk)]\,\vec{E}_0(k)A(\vec{r},t)\,dk=0$$

[a similar integral relation is found from (27)]. This will be identically satisfied for all $\vec{r}$ and t if

$$s^2-k^2+\varepsilon\mu\omega^2=0 \quad \text{and} \quad \mu\sigma\omega-2sk=0 \tag{33}$$

By using relations (33), ω and s can be expressed as functions of k, as required in order that the integral relations (28) make sense. Notice, in particular, that, by the second relation (33), s=0 if σ=0 (non-conducting medium). Then, by the first relation, $\omega/k=1/(\varepsilon\mu)^{1/2}$, which is the familiar expression for the speed of propagation of an e/m wave in a non-conducting medium [3].

From the two Gauss' laws (25*a*) and (25*b*) we get the corresponding integral relations

$$\int(ik-s)\,\hat{\tau}\cdot\vec{E}_0(k)A(\vec{r},t)\,dk=0\ ,$$
$$\int(ik-s)\,\hat{\tau}\cdot\vec{B}_0(k)A(\vec{r},t)\,dk=0\ .$$

These will be identically satisfied for all $\vec{r}$ and t if

$$\hat{\tau}\cdot\vec{E}_0(k)=0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B}_0(k)=0,\ \text{for all } k \tag{34}$$

From (25*c*) and (25*d*) we find

$$\int(ik-s)\,\hat{\tau}\times\vec{E}_0(k)A(\vec{r},t)\,dk=\int i\omega\vec{B}_0(k)A(\vec{r},t)\,dk$$

and

$$\int(ik-s)\,\hat{\tau}\times\vec{B}_0(k)A(\vec{r},t)\,dk=\int(\mu\sigma-i\varepsilon\mu\omega)\vec{E}_0(k)A(\vec{r},t)\,dk\ ,$$

respectively. To satisfy these for all $\vec{r}$ and t, we require that

$$(k+is)\,\hat{\tau}\times\vec{E}_0(k)=\omega\vec{B}_0(k) \tag{35}$$

and

$$(k+is)\,\hat{\tau}\times\vec{B}_0(k)=-(\varepsilon\mu\omega+i\mu\sigma)\vec{E}_0(k) \tag{36}$$

Note, however, that (36) is not an independent equation since it can be reproduced by cross-multiplying (35) with $\hat{\tau}$ and by taking into account Eqs. (33) and (34).

We note the following:

1. From (30) and (34) we have that

$$\hat{\tau}\cdot\vec{E}=0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B}=0 \tag{37}$$

or, in real form, $\hat{\tau}\cdot\mathrm{Re}\,\vec{E}=0$ and $\hat{\tau}\cdot\mathrm{Re}\,\vec{B}=0$. This means that both $\mathrm{Re}\,\vec{E}$ and $\mathrm{Re}\,\vec{B}$ are normal to the direction of propagation of the wave.

2. From (30) and (35) we get

$$\hat{\tau}\times\vec{E}=\int\frac{\omega}{k+is}\vec{B}_0(k)A(\vec{r},t)\,dk \tag{38}$$

The integral on the right-hand side of (38) is, generally, not a vector parallel to $\vec{B}$. Now, in the limit of negligible conductivity (σ=0) the relations (33) give s=0 and $\omega/k=1/(\varepsilon\mu)^{1/2}$. The ratio ω/k represents the speed of propagation υ in the non-conducting medium, for the frequency ω. If the medium is *non-dispersive*, the speed $\upsilon=\omega/k$ is constant, independent of frequency. Then Eq. (38) (with s=0) becomes

$$\hat{\tau}\times\vec{E}=\upsilon\int\vec{B}_0(k)A(\vec{r},t)\,dk=\upsilon\vec{B}$$

and, in real form, it reads $\hat{\tau}\times\mathrm{Re}\vec{E}=\upsilon\,\mathrm{Re}\vec{B}$. Geometrically, this means that the $(\mathrm{Re}\,\vec{E},\mathrm{Re}\,\vec{B},\hat{\tau})$ define a right-handed rectangular system.

3. As shown in [7,8], the $\vec{E}$ and $\vec{B}$ are always mutually perpendicular in a *monochromatic* e/m wave of definite frequency ω, traveling in a conducting medium. Such a wave is represented in real form by the equations

$$\vec{E}(\vec{r},t)=\vec{E}_0\,e^{-s\hat{\tau}\cdot\vec{r}}\cos(k\hat{\tau}\cdot\vec{r}-\omega t+\alpha)\ ,$$
$$\vec{B}(\vec{r},t)=\frac{\sqrt{k^2+s^2}}{\omega}(\hat{\tau}\times\vec{E}_0)\,e^{-s\hat{\tau}\cdot\vec{r}}\cos(k\hat{\tau}\cdot\vec{r}-\omega t+\beta)$$

where $\vec{E}_0$ is a real vector and where $\tan(\beta-\alpha)=s/k$. This perpendicularity between $\vec{E}$ and $\vec{B}$ ceases to exist, however, in a non-monochromatic wave of the form (28).

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Independence of Maxwell's equations: A Bäcklund-transformation view

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Abstract. It is now widely accepted that the Maxwell equations of Electrodynamics constitute a self-consistent set of four independent partial differential equations. According to a certain school of thought, however, half of these equations – namely, those expressing the two Gauss' laws for the electric and the magnetic field – are redundant since they can be "derived" from the remaining two laws and the principle of conservation of charge. The status of the latter principle is thus elevated to a law of Nature more fundamental than, say, Coulomb's law. In this note we examine this line of reasoning and we propose an approach according to which the Maxwell equations may be viewed as a Bäcklund transformation relating fields and sources. The conservation of charge and the electromagnetic wave equations then simply express the integrability conditions of this transformation.

Keywords: Classical electrodynamics, Maxwell's equations, Bäcklund transformations

1. Is Gauss' law of Electrodynamics redundant?

As we know, the *Maxwell equations* describe the behavior (that is, the laws of change in space and time) of the electromagnetic (e/m) field. This field is represented by the pair $(\vec{E},\vec{B})$, where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. The Maxwell equations additionally impose certain boundary conditions at the interface of two different media, while certain other physical demands are obvious (for example, the e/m field must vanish away from its localized "sources", unless these sources emit e/m radiation).

The Maxwell equations are a system of four partial differential equations (PDEs) that is self-consistent, in the sense that these equations are compatible with one another. The self-consistency of the system also implies the satisfaction of two important conditions that are physically meaningful:

- the *equation of continuity*, related to *conservation of charge*; and
- the *e/m wave equation* in its various forms.

We stress that these conditions are *necessary but not sufficient* for the validity of the Maxwell system. Thus, although every solution $(\vec{E},\vec{B})$ of this system obeys a wave equation separately for the electric and the magnetic field, an arbitrary pair of fields

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$(\vec{E},\vec{B})$, each field satisfying the corresponding wave equation, does not necessarily satisfy the Maxwell system itself. Also, the principle of conservation of charge *cannot replace* any one of Maxwell's equations. These remarks are justified by the fact that the aforementioned two necessary conditions are derived by *differentiating* the Maxwell system and, in this process, part of the information carried by this system is lost. [Recall, similarly, that cross-differentiation of the Cauchy-Riemann relations of complex analysis yields the Laplace equation (see Sec. 2) by which, however, we cannot recover the Cauchy-Riemann relations.]

The differential form of the Maxwell equations is

$$\begin{aligned} &(a) \quad \vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} && (c) \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b) \quad \vec{\nabla}\cdot\vec{B}=0 && (d) \quad \vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{1}$$

where $\rho, \vec{J}$ are the charge and current densities, respectively (the "sources" of the e/m field). Both the fields and the sources are functions of the spacetime variables (x,y,z,t). Equations (1*a*) and (1*b*), which describe the *div* of the e/m field at any moment, constitute *Gauss' law* for the electric and the magnetic field, respectively. In terms of physical content, (1*a*) expresses the Coulomb law of electricity, while (1*b*) rules out the possibility of existence of magnetic poles analogous to electric charges. Equation (1*c*) expresses the *Faraday-Henry law* (law of e/m induction) and Eq. (1*d*) expresses the *Ampère-Maxwell law*. Equations (1*a*) and (1*d*), which contain the sources of the e/m field, constitute the *non-homogeneous* Maxwell equations, while Eqs. (1*b*) and (1*c*) are the *homogeneous* equations of the system.

By taking the *div* of (1*d*) and by using (1*a*), we obtain the *equation of continuity*, which physically expresses the *principle of conservation of charge* (see, e.g., [1], Sec. 9.6):

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0 \tag{2}$$

Although the charge and current densities on the right-hand sides of (1*a*) and (1*d*) are chosen freely and are considered known from the outset, relation (2) places a severe restriction on the associated functions. A different kind of differentiation of the Maxwell system (1), by taking the *rot* of (*c*) and (*d*), leads to separate wave equations (or modified wave equations, depending on the medium) for the electric and the magnetic field (see, e.g., [1], Sec. 10.4).

In most textbooks on electromagnetism (e.g., [2–6] and many more) the Maxwell equations (1) are treated as a consistent set of four independent PDEs. A number of authors, however, have doubted the independence of this system. Specifically, they argue that (1*a*) and (1*b*) – the equations for the *div* of the e/m field, expressing Gauss' law for the corresponding fields – are redundant since they "may be derived" from (1*c*) and (1*d*) in combination with the equation of continuity (2). If this is true, Coulomb's law – the most important experimental law of electricity – loses its status as an independent law and is reduced to a derivative theorem. The same can be said with regard to the non-existence of magnetic poles in Nature.

As far as we know, the first who doubted the independent status of the two Gauss' laws in electrodynamics was Julius Adams Stratton in his 1941 famous (and, admittedly, very attractive) book [7]. His reasoning may be described as follows:

By taking the *div* of (1*c*), the left-hand side vanishes identically while on the right-hand side we may change the order of differentiation with respect to space and time variables. The result is:

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{B}\right)=0 \tag{3}$$

On the other hand, by taking the *div* of (1*d*) and by using the equation of continuity (2), we find that

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{E}-\frac{\rho}{\varepsilon_0}\right)=0 \tag{4}$$

And the line of argument continues as follows: According to (3) and (4), the quantities $\vec{\nabla}\cdot\vec{B}$ and ($\vec{\nabla}\cdot\vec{E}-\rho/\varepsilon_0$) are constant in time at every point (*x*,*y*,*z*) of the region *Ω* of space that concerns us. *If* we now assume that there has been a period of time during which no e/m field existed in the region *Ω*, then, in that period,

$$\vec{\nabla}\cdot\vec{B}=0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{E}-\rho/\varepsilon_0=0 \tag{5}$$

identically. Later on, although an e/m field did appear in *Ω*, the left-hand sides in (5) continued to vanish everywhere within this region since, as we said above, those quantities are time constant at every point of *Ω*. Thus, by the equations for the *rot* of the e/m field and by the principle of conservation of charge – the status of which was elevated from derivative theorem to fundamental law of the theory – we derived Eqs. (5), which are precisely the first two Maxwell equations (1*a*) and (1*b*)!

According to this reasoning, the electromagnetic theory is not based on four independent Maxwell equations but rather on *three* independent equations only; namely, the Faraday-Henry law (1*c*), the Ampère–Maxwell law (1*d*), and the principle of conservation of charge (2).

What makes this view questionable is the assumption that, for *every* region *Ω* of space there exists some period of time during which the e/m field in *Ω* vanishes. This hypothesis is arbitrary and is not dictated by the theory itself. (It is likely that no such region exists in the Universe!) Therefore, the argument that led from relations (3) and (4) to relations (5) is not convincing since it was based on an arbitrary and, in a sense, artificial initial condition: that the e/m field is zero at some time t=0 and before.

Let us assume for the sake of argument, however, that there exists a region *Ω* within which the e/m field is zero for $t<t_0$ and nonzero for $t>t_0$. The critical issue is what happens at $t=t_0$; specifically, whether the functions expressing the e/m field are *continuous* at that moment. If they indeed are, the field starts from zero and gradually increases to nonzero values; thus, the line of reasoning that led from (3) and (4) to (5) is acceptable. There are physical situations, however, in which the appearance of an e/m field is so abrupt that it may be considered *instantaneous*. (For instance, the moment we connect the ends of a metal wire to a battery, an electric field suddenly appears in the interior of the wire and a magnetic field appears in the exterior. An

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even more "dramatic" example is *pair production* in which a charged particle and the corresponding antiparticle are created simultaneously, thus an e/m field appears at that moment in the region.) In such cases the e/m field is *non-continuous* at t=t_0 and its time derivative is *not defined* at this instant. Therefore, the line of reasoning that leads from (3) and (4) to (5) again collapses.

Note, finally, a circular reasoning in Stratton's approach. It is assumed that, in a region Ω where no e/m field exists, the second of relations (5) is valid identically. This means that the vanishing of the electric field in Ω automatically implies the absence of electric charge in that region. This fact, however, follows from Gauss' law (1*a*); thus it may not be used *a priori* as a tool for proving the law itself!

Regarding charge conservation, we mentioned earlier that Eq. (2) is derived from the two non-homogeneous Maxwell equations, namely, Gauss' law (1*a*) for the electric field, and the Ampère–Maxwell law (1*d*). This means that the principle of conservation of charge is a *necessary* condition in order for the Maxwell system to be self-consistent. This condition is *not sufficient*, however, in the sense that it cannot replace any one of the system equations. Indeed, by the Ampère–Maxwell law and the conservation of charge there follows the *time derivative* of Gauss' law for the electric field [Eq. (4)]; this, however, does not imply that Gauss' law itself is valid. Of course, the reverse is true: *because* Gauss' law is valid, the same is true for its time derivative.

Our view, therefore, is that the Maxwell equations form a system of four independent PDEs that express respective laws of Nature. Moreover, the self-consistency of this system imposes two *necessary* (but *not sufficient*) conditions that concern the conservation of charge and the wave behavior of the time-dependent e/m field. In the next section the problem is re-examined from the point of view of Bäcklund transformations.

2. A Bäcklund-transformation view of Maxwell's equations

In previous articles [8,9] we suggested that, mathematically speaking, the Maxwell equations in empty space may be viewed as a Bäcklund transformation (BT) relating the electric and the magnetic field to each other. Let us briefly summarize a few key points regarding this idea. To begin with, let us see the simplest, perhaps, example of a BT.

The *Cauchy-Riemann relations* of complex analysis,

$$u_x = v_y \quad (a) \qquad u_y = -v_x \quad (b) \tag{6}$$

(where subscripts denote partial derivatives with respect to the indicated variables) constitute a BT for the *Laplace equation*,

$$w_{xx} + w_{yy} = 0 \tag{7}$$

Let us explain this: Suppose we want to solve the system (6) for u, for a given choice of the function $v(x,y)$. To see if the PDEs (6*a*) and (6*b*) match for solution for u, we must compare them in some way. We thus differentiate (6*a*) with respect to y and (6*b*) with respect to x, and equate the mixed derivatives of u. That is, we apply the *integrability condition* (or *consistency condition*) $(u_x)_y = (u_y)_x$. In this way we eliminate the variable u and we find a condition that must be obeyed by $v(x,y)$:

$$v_{xx} + v_{yy} = 0.$$

Similarly, by using the integrability condition $(v_x)_y = (v_y)_x$ to eliminate v from the system (6), we find the necessary condition in order that this system be integrable for v, for a given function $u(x,y)$:

$$u_{xx} + u_{yy} = 0.$$

In conclusion, the integrability of system (6) with respect to either variable requires that the other variable satisfy the Laplace equation (7).

Let now $v_0(x,y)$ be a known solution of the Laplace equation (7). Substituting $v=v_0$ in the system (6), we can integrate this system with respect to u. It is not hard to show (by eliminating v_0 from the system) that the solution u will also satisfy the Laplace equation. As an example, by choosing the solution $v_0(x,y)=xy$ of (7), we find a new solution $u(x,y)=(x^2-y^2)/2+C$.

Generally speaking, a BT is a system of PDEs connecting two functions (say, u and v) in such a way that the consistency of the system requires that u and v independently satisfy the respective, higher-order PDEs $F[u]=0$ and $G[v]=0$. Analytically, in order that the system be integrable for u, the function v must be a solution of $G[v]=0$; conversely, in order that the system be integrable for v, the function u must be a solution of $F[u]=0$. If F and G happen to be functionally identical, as in the example given above, the BT is said to be an *auto-Bäcklund* transformation (auto-BT).

Classically, BTs are useful tools for finding solutions of nonlinear PDEs. In [8,9], however, we suggested that BTs may also be useful for solving *linear systems* of PDEs. The prototype example that we used was the Maxwell equations in empty space:

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E}=0 \qquad &(c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B}=0 \qquad &(d)\quad \vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{8}$$

Here we have a system of four PDEs for two vector fields that are functions of the spacetime coordinates (x,y,z,t). We would like to find the integrability conditions necessary for self-consistency of the system (8). To this end, we try to uncouple the system to find separate second-order PDEs for $\vec{E}$ and $\vec{B}$, the PDE for each field being a necessary condition in order that the system (8) be integrable for the other field. This uncoupling, which eliminates either field (electric or magnetic) in favor of the other, is achieved by properly differentiating the system equations and by using suitable vector identities, in a manner similar in spirit to that which took us from the first-order Cauchy-Riemann system (6) to the separate second-order Laplace equations (7) for u and v.

As discussed in [8,9], the only nontrivial integrability conditions for the system (8) are those obtained by using the vector identities

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \tag{9}$$

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$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \tag{10}$$

By these we obtain separate wave equations for the electric and the magnetic field:

$$\nabla^2\vec{E}-\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}=0 \tag{11}$$

$$\nabla^2\vec{B}-\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}=0 \tag{12}$$

We conclude that the Maxwell system (8) in empty space is a BT relating the e/m wave equations for the electric and the magnetic field, in the sense that the wave equation for each field is an integrability condition for solution of the system in terms of the other field.

The case of the full Maxwell equations (1) is more complex due to the presence of the source terms $\rho, \vec{J}$ in the non-homogeneous equations (1*a*) and (1*d*). As it turns out, the self-consistency of the BT imposes restrictions on the terms of non-homogeneity as well as on the fields themselves. Before we get to this, however, let us see a simpler "toy" example that generalizes that of the Cauchy-Riemann relations.

Consider the following non-homogeneous linear system of PDEs for the functions u and v of the variables x, y, z, t:

$$\begin{aligned} &u_x = v_y \quad (a) \qquad && u_z = v_z + p(x,y,z,t) \quad (c) \\ &u_y = -v_x \quad (b) \qquad && u_t = v_t + q(x,y,z,t) \quad (d) \end{aligned} \tag{13}$$

where p and q are assumed to be given functions. The necessary consistency conditions for this system are found by cross-differentiation of the system equations with respect to the variables x, y, z, t. In particular, by cross-differentiating (*a*) and (*b*) with respect to x and y we find that $u_{xx}+u_{yy}=0$ and $v_{xx}+v_{yy}=0$; hence both u and v must satisfy the Laplace equation (7). On the other hand, cross-differentiation of (*c*) and (*d*) with respect to z and t eliminates the fundamental variables u and v, yielding a necessary condition for the terms of non-homogeneity, p and q; that is, $p_t - q_z = 0$. This means that the functions p and q cannot be chosen arbitrarily from the outset but must conform to this latter condition in order for the system (13) to have a solution.

As an application, let us take $v=xy+zt$ (which satisfies the Laplace equation $v_{xx}+v_{yy}=0$) and let us choose $p=2t$ and $q=2z$ (so that $p_t - q_z = 0$). It is not hard to show that the solution of the system (13) for u is then given by

$$u(x,y,z,t) = (x^2-y^2)/2+3zt+C\,.$$

Notice that $u_{xx}+u_{yy}=0$, as expected.

Let us now return to the full Maxwell equations (1), which we now view as a BT relating the electric and the magnetic field and containing additional terms in which only the sources appear. As can be checked, there are now three nontrivial integrability conditions, namely, those found by applying the vector identities (9) and (10), as well as the identity

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right) = 0 \tag{14}$$

(the corresponding one for $\vec{E}$ is trivially satisfied in view of the Maxwell system). By (9) and (10) we get the non-homogeneous wave equations

$$\nabla^2 \vec{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\varepsilon_0} \vec{\nabla} \rho + \mu_0 \frac{\partial \vec{J}}{\partial t} \tag{15}$$

$$\nabla^2 \vec{B} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_0 \vec{\nabla} \times \vec{J} \tag{16}$$

Additionally, the integrability condition (14) yields the equation of continuity (2),

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{17}$$

expressing conservation of charge. Notice that, unlike (15) and (16), the condition (17) places *a priori* restrictions *on the sources* rather than on the fields themselves!

In any case, the three relations (15) – (17) are *necessary* conditions imposed by the requirement of self-consistency of the BT (1). As explained in Sec. 1, however, these conditions are *not sufficient*, in the sense that none of them may replace any equation in the system (1). In particular, the equation of continuity (17) may not be regarded as more fundamental than the Gauss law (1*a*) for the electric field.

3. Conclusions

Let us summarize our main conclusions:

1. The Maxwell equations (1) express four separate laws of Nature. These equations are mathematically consistent with one another but constitute a set of independent vector relations, in the sense that no single equation may be deduced by the remaining three. In particular, the physical arguments that attempt to render the two Gauss' laws "redundant" are seen to be artificial and unrealistic.

2. We consider the Maxwell equations as physically acceptable simply because the system (1) and all conclusions mathematically drawn from it represent experimentally verifiable situations in Nature. Among these conclusions are the conservation of charge and the conservation of energy (Poynting's theorem). It should be kept in mind, however, that conservation laws appear as *consequences* of the fundamental equations of a theory, and not vice versa. In particular, conservation of charge, in the form of the continuity equation (17), is a physically verifiable mathematical conclusion drawn from the Maxwell system (1) but it may not be regarded as more fundamental than any equation in the system. The same can be said with regard to the existence of e/m waves, expressed mathematically by Eqs. (11) and (12).

3. From a mathematical perspective, the Maxwell system (1) may be viewed as a Bäcklund transformation (BT) the integrability conditions of which (i.e., the *necessary* conditions for self-consistency of the system) yield separate (generally non-

homogeneous) wave equations (15) and (16) for the electric and the magnetic field, respectively, as well as the equation of continuity (17). These integrability conditions are derived by *differentiating* the BT in different ways; hence they carry less information than the BT itself. Consequently, none of the integrability conditions may replace any equation in the Maxwell system.

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Infinitesimal symmetry transformations of matrix-valued differential equations: An algebraic approach

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Abstract. The study of symmetries of partial differential equations (PDEs) has been traditionally treated as a geometrical problem. Although geometrical methods have been proven effective with regard to finding infinitesimal symmetry transformations, they present certain conceptual difficulties in the case of matrix-valued PDEs; for example, the usual differential-operator representation of the symmetry-generating vector fields is not possible in this case. An algebraic approach to the symmetry problem of PDEs is described, based on abstract operators (characteristic derivatives) which admit a standard differential-operator representation in the case of scalar-valued PDEs.

Keywords: Matrix-valued differential equations, symmetry transformations, Lie algebras, recursion operators

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1. Introduction

The problem of symmetries of a system of partial differential equations (PDEs) has been traditionally treated as a geometrical problem in the jet space of the independent and the dependent variables (including a sufficient number of partial derivatives of the latter variables with respect to the former ones). Two more or less equivalent approaches have been followed: (*a*) invariance of the system of PDEs itself, under infinitesimal transformations generated by corresponding vector fields in the jet space [1]; (*b*) invariance of a differential ideal of differential forms representing the system of PDEs, under the Lie derivative with respect to the vector fields representing the symmetry transformations [2-6].

Although effective with regard to calculating symmetries, these geometrical approaches suffer from a certain drawback of conceptual nature when it comes to matrix-valued PDEs. The problem is related to the inevitably mixed nature of the coordinates in the jet space (scalar independent variables versus matrix-valued dependent ones) and the need for a differential-operator representation of the symmetry vector fields. How does one define differentiation with respect to matrix-valued variables? Moreover, how does one calculate the Lie bracket of two differential operators in which some (or all) of the variables, as well as the coefficients of partial derivatives with respect to these variables, are matrices?

Although these difficulties were handled in some way in [4-6], it was eventually realized that an alternative, purely algebraic approach to the symmetry problem would be more appropriate in the case of matrix-valued PDEs. Elements of this approach were presented in [7] and later

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applied in particular problems [8-10]; no formal theoretical framework was fully developed, however.

An attempt to develop such a framework is made in this article. In Sections 2 and 3 we introduce the concept of the *characteristic derivative* – an abstract operator generalization of the Lie derivative used in scalar-valued problems – and we demonstrate the Lie-algebraic nature of the set of these derivatives.

The general symmetry problem for matrix-valued PDEs is presented in Sec. 4, and the Lie-algebraic property of symmetries of a PDE is proven in Sec. 5. In Sec. 6 we discuss the concept of a *recursion operator* [1,8-14] by which an infinite set of symmetries may in principle be produced from any known symmetry.

Finally, an application of these ideas is made in Sec. 7 by using the chiral field equation as an example.

To simplify our formalism, we restrict our analysis to the case of a single matrix-valued PDE in one dependent variable. Generalization to systems of PDEs is straightforward and is left to the reader (see, e.g., [1] for scalar-valued problems).

2. Total and characteristic derivative operators

A PDE for the unknown function $u=u(x^1, x^2, \ldots) \equiv u(x^k)$ [where by (x^k) we collectively denote the independent variables $x^1, x^2, \ldots$] is an expression of the form $F[u]=0$, where $F[u] \equiv F(x^k, u, u_k, u_{kl}, \ldots)$ is a function in the *jet space* [1] of the independent variables (x^k), the dependent variable u, and the partial derivatives of various orders of u with respect to the x^k, which derivatives will be denoted by using subscripts: u_k, u_{kl}, u_{klm}, etc. A *solution* of the PDE is any function $u=\varphi(x^k)$ for which the relation $F[u]=0$ is satisfied at each point (x^k) in the domain of φ.

We assume that u, as well as all functions $F[u]$ in the jet space, are square-matrix-valued of fixed matrix dimensions. In particular, we require that, in its most general form, a function $F[u]$ in the jet space is expressible as a finite or an infinite sum of products of alternating x-dependent and u-dependent terms, of the form

$$F[u] = \sum a(x^k)\,\Pi[u]\,b(x^k)\,\Pi'[u]\,c(x^k)\cdots \tag{2.1}$$

where the $a(x^k)$, $b(x^k)$, $c(x^k)$, etc., are matrix-valued, and where the matrices $\Pi[u]$, $\Pi'[u]$, etc., are products of variables u, u_k, u_{kl}, etc., of the "fiber" space (or, more generally, products of powers of these variables). The set of all functions (2.1) is thus a (generally) non-commutative algebra.

If u is a scalar quantity, a total derivative operator can be defined in the usual way [1] as

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + u_{ijk} \frac{\partial}{\partial u_{jk}} + \cdots \tag{2.2}$$

where the summation convention over repeated up-and-down indices (such as j and k in this equation) has been adopted and will be used throughout. If, however, u is matrix-valued, the above expression is obviously not valid. A generalization of the definition of the total derivative is thus necessary for matrix-valued PDEs.

Definition 2.1. The *total derivative operator* with respect to the variable x^i is a linear operator D_i acting on functions $F[u]$ of the form (2.1) in the jet space and having the following properties:

1. On functions $f(x^k)$ in the base space, $D_i\, f(x^k) = \partial f / \partial x^i \equiv \partial_i\, f(x^k)$.

2. For $F[u]=u$, u_i, u_{ij}, etc., we have: $D_i\, u = u_i$, $D_i\, u_j = D_j\, u_i = u_{ij} = u_{ji}$, etc.

3. The operator D_i is a *derivation* on the algebra of all matrix-valued functions of the form (2.1) in the jet space; i.e., the *Leibniz rule* is satisfied:

$$D_i\left(F[u]G[u]\right)=\left(D_iF[u]\right)G[u]+F[u]D_iG[u] \tag{2.3}$$

Higher-order total derivatives $D_{ij}=D_iD_j$ may similarly be defined but they no longer possess the derivation property. Given that $\partial_i\partial_j=\partial_j\partial_i$ and that $u_{ij}=u_{ji}$, it follows that $D_iD_j = D_jD_i \Leftrightarrow D_{ij} = D_{ji}$; that is, total derivatives commute. We write: $[D_i, D_j]=0$, where, in general, $[A, B] \equiv AB–BA$ will denote the *commutator* of two operators or two matrices, as the case may be.

If u^{-1} is the inverse of u, such that $uu^{-1}= u^{-1}u = \mathbf{1}$, then we can define

$$D_i\left(u^{-1}\right)\equiv -u^{-1}\left(D_iu\right)u^{-1} \tag{2.4}$$

Moreover, for any functions $A[u]$ and $B[u]$ in the jet space, it can be shown that

$$D_i[A,B]=\left[D_iA,\, B\right]+\left[A,\, D_iB\right] \tag{2.5}$$

As an example, let $(x^1, x^2) \equiv (x, t)$ and let $F[u]=xtu_x^2$, where u is matrix-valued. Writing $F[u]=xtu_xu_x$, we have: $D_t\, F[u]=xu_x^2 + xt\,(u_{xt}\, u_x + u_x\, u_{xt})$.

Let now $Q[u] \equiv Q\,(x^k, u, u_k, u_{kl}, \ldots)$ be a function in the jet space. We will call this a *characteristic function* (or simply a *characteristic*) of a certain derivative, defined as follows:

Definition 2.2. The *characteristic derivative* with respect to $Q[u]$ is a linear operator Δ_Q acting on functions $F[u]$ in the jet space and having the following properties:

1. On functions $f(x^k)$ in the base space,

$$\Delta_Q\, f(x^k)=0 \tag{2.6}$$

(that is, Δ_Q acts only in the fiber space).

2. For $F[u]=u$,

$$\Delta_Q u=Q[u] \tag{2.7}$$

3. Δ_Q commutes with total derivatives:

$$\Delta_Q D_i = D_i\Delta_Q \;\Leftrightarrow\; [\Delta_Q, D_i]=0 \quad (\text{all } i) \tag{2.8}$$

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4. The operator Δ_Q is a *derivation* on the algebra of all matrix-valued functions of the form (2.1) in the jet space (the Leibniz rule is satisfied):

$$\Delta_Q\left(F[u]G[u]\right)=\left(\Delta_Q F[u]\right)G[u]+F[u]\Delta_Q G[u] \tag{2.9}$$

Corollary: By (2.7) and (2.8) we have:

$$\Delta_Q u_i=\Delta_Q D_i u=D_i Q[u] \tag{2.10}$$

We note that the operator Δ_Q is a well-defined quantity, in the sense that the action of Δ_Q on u uniquely determines the action of Δ_Q on any function $F[u]$ of the form (2.1) in the jet space. Moreover, since, by assumption, u and $Q[u]$ are matrices of equal dimensions, it follows from (2.7) that Δ_Q preserves the matrix character of u, as well as of any function $F[u]$ on which this operator acts.

We also remark that we have imposed conditions (2.6) and (2.8) having a certain property of symmetries of PDEs in mind; namely, *every* symmetry of a PDE can be represented by a transformation of the dependent variable u alone, i.e., can be expressed as a transformation in the fiber space (see [1], Chap. 5).

The following formulas, analogous to (2.4) and (2.5), may be written:

$$\Delta_Q\left(u^{-1}\right)\equiv -u^{-1}\left(\Delta_Q u\right)u^{-1} \tag{2.11}$$

$$\Delta_Q[A,B]=\left[\Delta_Q A,\,B\right]+\left[A,\,\Delta_Q B\right] \tag{2.12}$$

As an example, let $(x^1, x^2)\equiv(x, t)$ and let $F[u]=a(x,t)u^2b(x,t)+[u_x\, , u_t]$, where a, b and u are matrices of equal dimensions. Writing $u^2=uu$ and using properties (2.7), (2.9), (2.10) and (2.12), we find: $\Delta_Q F[u]=a(x,t)(Qu+uQ)b(x,t)+[D_x Q,\, u_t]+[u_x\, , D_t Q]$.

In the case where u is scalar-valued (thus so is $Q[u]$), the characteristic derivative Δ_Q admits a differential-operator representation of the form

$$\Delta_Q=Q[u]\frac{\partial}{\partial u}+\left(D_i Q[u]\right)\frac{\partial}{\partial u_i}+\left(D_i D_j Q[u]\right)\frac{\partial}{\partial u_{ij}}+\cdots \tag{2.13}$$

(See [1], Chap. 5, for an analytic proof of property (2.8) in this case.)

3. The Lie algebra of characteristic derivatives

The characteristic derivatives Δ_Q acting on functions $F[u]$ of the form (2.1) in the jet space constitute a *Lie algebra of derivations* on the algebra of the $F[u]$. The proof of this statement is contained in the following three Propositions.

Proposition 3.1. Let Δ_Q be a characteristic derivative with respect to the characteristic $Q[u]$; i.e., $\Delta_Q u=Q[u]$ [cf. Eq. (2.7)]. Let λ be a constant (real or complex). We define the operator $\lambda\Delta_Q$ by the relation

$$(\lambda\Delta_Q)F[u] \equiv \lambda(\Delta_Q F[u]) .$$

Then, $\lambda\Delta_Q$ is a characteristic derivative with characteristic $\lambda Q[u]$. That is,

$$\lambda\Delta_Q = \Delta_{\lambda Q} \tag{3.1}$$

Proof. (*a*) The operator $\lambda\Delta_Q$ is linear, since so is Δ_Q .

(*b*) For $F[u]=u$, $(\lambda\Delta_Q)u=\lambda(\Delta_Q u)=\lambda Q[u]$.

(*c*) $\lambda\Delta_Q$ commutes with total derivatives D_i , since so does Δ_Q .

(*d*) Given that Δ_Q satisfies the Leibniz rule (2.9), it is easily shown that so does $\lambda\Delta_Q$.

Comment: Condition (*c*) would not be satisfied if we allowed λ to be a function of the x^k, instead of being a constant, since $\lambda(x^k)$ generally does not commute with the D_i. Therefore, relation (3.1) is not valid for a non-constant λ.

Proposition 3.2. Let Δ_1 and Δ_2 be characteristic derivatives with respect to the characteristics $Q_1[u]$ and $Q_2[u]$, respectively; i.e., $\Delta_1 u=Q_1[u]$, $\Delta_2 u=Q_2[u]$. We define the operator $\Delta_1+\Delta_2$ by

$$(\Delta_1+\Delta_2)F[u] \equiv \Delta_1 F[u] + \Delta_2 F[u] .$$

Then, $\Delta_1+\Delta_2$ is a characteristic derivative with characteristic $Q_1[u]+Q_2[u]$. That is,

$$\Delta_1+\Delta_2=\Delta_Q \quad \text{with} \quad Q[u]=Q_1[u]+Q_2[u] \tag{3.2}$$

Proof. (*a*) The operator $\Delta_1+\Delta_2$ is linear, as a sum of linear operators.

(*b*) For $F[u]=u$, $(\Delta_1+\Delta_2)u = \Delta_1 u+\Delta_2 u = Q_1[u]+Q_2[u]$.

(*c*) $\Delta_1+\Delta_2$ commutes with total derivatives D_i , since so do Δ_1 and Δ_2 .

(*d*) Given that each of Δ_1 and Δ_2 satisfies the Leibniz rule (2.9), it is not hard to show that the same is true for $\Delta_1+\Delta_2$.

Proposition 3.3. Let Δ_1 and Δ_2 be characteristic derivatives with respect to the characteristics $Q_1[u]$ and $Q_2[u]$, respectively; i.e., $\Delta_1 u=Q_1[u]$, $\Delta_2 u=Q_2[u]$. We define the operator $[\Delta_1 , \Delta_2]$ (*Lie bracket* of Δ_1 and Δ_2) by

$$[\Delta_1 , \Delta_2]F[u] \equiv \Delta_1(\Delta_2 F[u]) - \Delta_2(\Delta_1 F[u]) .$$

Then, $[\Delta_1 , \Delta_2]$ is a characteristic derivative with characteristic $\Delta_1 Q_2[u]-\Delta_2 Q_1[u]$. That is,

$$[\Delta_1, \Delta_2]=\Delta_Q \quad \text{with} \quad Q[u]=\Delta_1 Q_2[u]-\Delta_2 Q_1[u] \equiv Q_{1,2}[u] \tag{3.3}$$

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Proof. (*a*) The linearity of $[\Delta_1, \Delta_2]$ follows from the linearity of Δ_1 and Δ_2.

(*b*) For $F[u]=u$, $[\Delta_1, \Delta_2]u = \Delta_1(\Delta_2 u) - \Delta_2(\Delta_1 u) = \Delta_1 Q_2[u] - \Delta_2 Q_1[u] \equiv Q_{1,2}[u]$.

(*c*) $[\Delta_1, \Delta_2]$ commutes with total derivatives D_i, since so do Δ_1 and Δ_2.

(*d*) Given that each of Δ_1 and Δ_2 satisfies the Leibniz rule (2.9), one can show (after some algebra and cancellation of terms) that the same is true for $[\Delta_1, \Delta_2]$.

In the case where u (thus the Q's also) is scalar-valued, the Lie bracket admits a standard differential-operator representation [1]:

$$[\Delta_1, \Delta_2] = Q_{1,2}[u]\frac{\partial}{\partial u} + \left(D_i Q_{1,2}\right)\frac{\partial}{\partial u_i} + \left(D_i D_j Q_{1,2}\right)\frac{\partial}{\partial u_{ij}} + \cdots \qquad (3.4)$$

where $Q_{1,2}[u] = [\Delta_1, \Delta_2]u = \Delta_1 Q_2[u] - \Delta_2 Q_1[u]$.

The Lie bracket $[\Delta_1, \Delta_2]$ has the following properties:

1. $[\Delta_1, a\Delta_2 + b\Delta_3] = a[\Delta_1, \Delta_2] + b[\Delta_1, \Delta_3]$;
   $[a\Delta_1 + b\Delta_2, \Delta_3] = a[\Delta_1, \Delta_3] + b[\Delta_2, \Delta_3]$ $\quad$ (a, b = const.)
2. $[\Delta_1, \Delta_2] = -[\Delta_2, \Delta_1]$ $\quad$ (*antisymmetry*)
3. $[\Delta_1, [\Delta_2, \Delta_3]] + [\Delta_2, [\Delta_3, \Delta_1]] + [\Delta_3, [\Delta_1, \Delta_2]] = 0$;
   $[[\Delta_1, \Delta_2], \Delta_3] + [[\Delta_2, \Delta_3], \Delta_1] + [[\Delta_3, \Delta_1], \Delta_2] = 0$ $\quad$ (*Jacobi identity*)

4. The symmetry problem for PDEs

Let $F[u]=0$ be a PDE in the independent variables $x^k \equiv x^1, x^2, \ldots$, and the (generally) matrix-valued dependent variable u. A transformation $u(x^k) \rightarrow u'(x^k)$ from the function u to a new function u' represents a *symmetry* of the PDE if the following condition is satisfied: $u'(x^k)$ is a solution of $F[u]=0$ *when* $u(x^k)$ is a solution; that is, $F[u']=0$ when $F[u]=0$.

We will restrict our attention to *continuous symmetries*, which can be expressed as infinitesimal transformations. Although such symmetries may involve transformations of the independent variables (x^k), they may equivalently be expressed as transformations of u alone (see [1], Chap. 5), i.e., as transformations in the fiber space.

An infinitesimal symmetry transformation is written symbolically as

$$u \rightarrow u' = u + \delta u$$

where δu is an infinitesimal quantity, in the sense that all powers $(\delta u)^n$ with $n>1$ may be neglected. The *symmetry condition* is thus written

$$F[u+\delta u] = 0 \quad \text{when} \quad F[u] = 0 \qquad (4.1)$$

An infinitesimal change δu of u induces a change $\delta F[u]$ of $F[u]$, where

$$\delta F[u] = F[u+\delta u] - F[u] \quad \Leftrightarrow \quad F[u+\delta u] = F[u] + \delta F[u] \tag{4.2}$$

Now, if δu is an infinitesimal symmetry and if u is a solution of $F[u]=0$, then $u+\delta u$ also is a solution; that is, $F[u+\delta u]=0$. This means that $\delta F[u]=0$ when $F[u]=0$. The symmetry condition (4.1) is thus written as follows:

$$\delta F[u] = 0 \mod F[u] \tag{4.3}$$

A symmetry transformation (we denote it M) of the PDE $F[u]=0$ produces a one-parameter family of solutions of the PDE from any given solution $u(x^k)$. We express this by writing

$$M: u(x^k) \to \bar{u}(x^k;\alpha) \quad \text{with} \quad \bar{u}(x^k;0) = u(x^k) \tag{4.4}$$

For infinitesimal values of the parameter α,

$$\bar{u}(x^k;\alpha) \simeq u(x^k) + \alpha Q[u] \quad \text{where} \quad Q[u] = \left.\frac{d\bar{u}}{d\alpha}\right|_{\alpha=0} \tag{4.5}$$

The function $Q[u] \equiv Q(x^k, u, u_k, u_{kl}, \ldots)$ in the jet space is called the *characteristic* of the symmetry (or, the *symmetry characteristic*). Putting

$$\delta u = \bar{u}(x^k;\alpha) - u(x^k) \tag{4.6}$$

we write, for infinitesimal α,

$$\delta u = \alpha Q[u] \tag{4.7}$$

We notice that the infinitesimal operator δ has the following properties:

1. According to its definition (4.2), δ is a linear operator:

$$\delta(F[u]+G[u]) = (F[u+\delta u]+ G[u+\delta u]) - (F[u]+G[u]) = \delta F[u]+\delta G[u] .$$

2. By assumption regarding the nature of our symmetry transformations, δ produces changes in the fiber space while it doesn't affect functions $f(x^k)$ in the base space [this is implicitly stated in (4.6)].

3. Since δ represents a difference, it commutes with all total derivatives D_i:

$$\delta(D_i A[u]) = D_i(\delta A[u]) .$$

In particular, for $A[u]=u$,

$$\delta u_i = \delta (D_i u) = D_i (\delta u) = \alpha D_i Q[u] ,$$

where we have used (4.7).

4. Since δ expresses an infinitesimal change, it may be approximated to a differential; in particular, it satisfies the Leibniz rule:

$$\delta (A[u]B[u]) = (\delta A[u]) B[u] + A[u] \delta B[u] .$$

For example, $\delta(u^2) = \delta(uu) = (\delta u)u + u\delta u = \alpha (Qu + uQ)$.

Now, consider the characteristic derivative Δ_Q with respect to the symmetry characteristic $Q[u]$. According to (2.7),

$$\Delta_Q u = Q[u] \tag{4.8}$$

We observe that the infinitesimal operator δ and the characteristic derivative Δ_Q share common properties. From (4.7) and (4.8) it follows that the two linear operators are related by

$$\delta u = \alpha \Delta_Q u \tag{4.9}$$

and, by extension,

$$\delta u_i = \alpha D_i Q[u] = \alpha \Delta_Q u_i , \text{ etc.}$$

[see (2.10)]. Moreover, for scalar-valued u and by the infinitesimal character of the operator δ, we may write:

$$\delta F[u] = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u_i} \delta u_i + \cdots = \alpha \left(\frac{\partial F}{\partial u} Q[u] + \frac{\partial F}{\partial u_i} D_i Q[u] + \frac{\partial F}{\partial u_{ij}} D_i D_j Q[u] + \cdots \right)$$

while, by (2.13),

$$\Delta_Q F[u] = \frac{\partial F}{\partial u} Q[u] + \frac{\partial F}{\partial u_i} D_i Q[u] + \frac{\partial F}{\partial u_{ij}} D_i D_j Q[u] + \cdots \tag{4.10}$$

The above observations lead us to the conclusion that, in general, the following relation is true:

$$\delta F[u] = \alpha \Delta_Q F[u] \tag{4.11}$$

The symmetry condition (4.3) is thus written:

$$\Delta_Q F[u] = 0 \mod F[u] \tag{4.12}$$

In particular, if u is scalar-valued, the above condition is written

$$\frac{\partial F}{\partial u}Q[u]+\frac{\partial F}{\partial u_i}D_i Q[u]+\frac{\partial F}{\partial u_{ij}}D_i D_j Q[u]+\cdots=0 \quad \mathrm{mod} \quad F[u] \tag{4.13}$$

which is a linear PDE for $Q[u]$. More generally, for matrix-valued u and for a function $F[u]$ of the form (2.1), the symmetry condition for the PDE $F[u]=0$ is a *linear* PDE for the symmetry characteristic $Q[u]$. We write this PDE symbolically as

$$S(Q;u) \equiv \Delta_Q F[u]=0 \quad \mathrm{mod} \quad F[u] \tag{4.14}$$

where the function $S(Q;u)$ is linear in Q and all total derivatives of Q. (The linearity of S in Q follows from the Leibniz rule and the specific form (2.1) of $F[u]$.)

Below is a list of formulas that may be useful in calculations:

- $\Delta_Q u_i = D_i Q[u]$, $\Delta_Q u_{ij} = D_i D_j Q[u]$, etc.
- $\Delta_Q u^2 = \Delta_Q (uu) = Q[u]u+uQ[u]$ (etc.)
- $\Delta_Q (u^{-1}) = -u^{-1} (\Delta_Q u) u^{-1} = -u^{-1} Q[u] u^{-1}$
- $\Delta_Q [A[u], B[u]] = [\Delta_Q A, B] + [A, \Delta_Q B]$

Comment: According to (4.12), $\Delta_Q F[u]$ vanishes if $F[u]$ vanishes. Given that Δ_Q is a linear operator, the reader may wonder whether this condition is identically satisfied (a linear operator acting on a zero function always produces a zero function!). Note, however, that the function $F[u]$ is *not identically* zero; it becomes zero only *for solutions* of the given PDE. What we need to do, therefore, is to first evaluate $\Delta_Q F[u]$ for *arbitrary* u and *then* demand that the result vanish *when* u is a solution of the PDE $F[u]=0$.

An alternative – and perhaps more transparent – version of the symmetry condition (4.12) is the requirement that the following relation be satisfied:

$$\Delta_Q F[u]=\hat{L}\,F[u] \tag{4.15}$$

where $\hat{L}$ is a linear operator acting on functions in the jet space (see, e.g., [1], Chap. 2 and 5, for a rigorous justification of this condition in the case of scalar-valued PDEs). For example, one may have

$$\Delta_Q F[u]=\sum_i \beta_i(x^k)D_i F[u]+\sum_{i,j}\gamma_{ij}(x^k)D_i D_j F[u]+A(x^k)F[u]+F[u]B(x^k)$$

where the β_i and γ_{ij} are scalar-valued, while A and B are matrix-valued. Let us see some examples, restricting for the moment our attention to scalar PDEs.

Example 4.1. The *sine-Gordon (s-G) equation* is written

$$F[u] \equiv u_{xt} - \sin u = 0 \,.$$

Here, $(x^1, x^2) \equiv (x, t)$. Since $\sin u$ can be expanded into an infinite series in powers of u, we see that $F[u]$ has the required form (2.1). Moreover, since u is a scalar function, we can write the symmetry condition by using (4.13):

$$S(Q;u) \equiv Q_{xt} - (\cos u)\, Q = 0 \quad \text{mod} \quad F[u]$$

where $S(Q;u) = \Delta_Q F[u]$ and where by subscripts we denote total differentiations with respect to the indicated variables. Let us verify the solution $Q[u] = u_x$. This characteristic corresponds to the symmetry transformation [cf. Eq. (4.4)]

$$M:\ u(x,t) \rightarrow \bar{u}(x,t;\alpha) = u(x+\alpha,\, t) \tag{4.16}$$

which implies that, if $u(x,t)$ is a solution of the s-G equation, then $\bar{u}(x,t) = u(x+\alpha,\, t)$ also is a solution. We have:

$$Q_{xt} - (\cos u)\, Q = (u_x)_{xt} - (\cos u)\, u_x = (u_{xt} - \sin u)_x = D_x F[u] = 0 \quad \text{mod} \quad F[u]\ .$$

Notice that $\Delta_Q F[u]$ is of the form (4.15), with $\hat{L} \equiv D_x$. Similarly, the characteristic $Q[u] = u_t$ corresponds to the symmetry

$$M:\ u(x,t) \rightarrow \bar{u}(x,t;\alpha) = u(x,\, t+\alpha) \tag{4.17}$$

That is, if $u(x,t)$ is a solution of the s-G equation, then so is $\bar{u}(x,t) = u(x,\, t+\alpha)$. The symmetries (4.16) and (4.17) reflect the fact that the s-G equation does not contain the variables x and t explicitly. (Of course, this equation has many more symmetries which are not displayed here; see, e.g., [1].)

Example 4.2. The *heat equation* is written

$$F[u] \equiv u_t - u_{xx} = 0\ .$$

The symmetry condition (4.13) reads

$$S(Q;u) \equiv Q_t - Q_{xx} = 0 \quad \text{mod} \quad F[u]$$

where $S(Q;u) = \Delta_Q F[u]$. As is easy to show, the symmetries (4.16) and (4.17) are valid here, too. Let us now try the solution $Q[u] = u$. We have:

$$Q_t - Q_{xx} = u_t - u_{xx} = F[u] = 0 \quad \text{mod} \quad F[u]\ .$$

This symmetry corresponds to the transformation

$$M:\ u(x,t) \rightarrow \bar{u}(x,t;\alpha) = e^{\alpha} u(x,t) \tag{4.18}$$

and is a consequence of the linearity of the heat equation.

Example 4.3. One form of the *Burgers equation* is

$$F[u] \equiv u_t - u_{xx} - u_x^{\,2} = 0 \,.$$

The symmetry condition (4.13) is written

$$S(Q;u) \equiv Q_t - Q_{xx} - 2u_x Q_x = 0 \quad \text{mod} \quad F[u]$$

where, as always, $S(Q;u)=\Delta_Q F[u]$. Putting $Q= u_x$ and $Q= u_t$, we verify the symmetries (4.16) and (4.17):

$$Q_t - Q_{xx} - 2u_x Q_x = u_{xt} - u_{xxx} - 2u_x u_{xx} = D_x F[u] = 0 \quad \text{mod} \quad F[u]$$
$$Q_t - Q_{xx} - 2u_x Q_x = u_{tt} - u_{xxt} - 2u_x u_{xt} = D_t F[u] = 0 \quad \text{mod} \quad F[u]$$

Note again that $\Delta_Q F[u]$ is of the form (4.15), with $\hat{L} \equiv D_x$ and $\hat{L} \equiv D_t$. Another symmetry is $Q[u]=1$, which corresponds to the transformation

$$M: \; u(x,t) \to \bar{u}(x,t;\alpha) = u(x,t) + \alpha \tag{4.19}$$

and is a consequence of the fact that u enters $F[u]$ only through its derivatives.

Example 4.4. The *wave equation* is written

$$F[u] \equiv u_{tt} - c^2 u_{xx} = 0 \quad (c = \text{const.})$$

and its symmetry condition reads

$$S(Q;u) \equiv Q_{tt} - c^2 Q_{xx} = 0 \quad \text{mod} \quad F[u] \,.$$

The solution $Q[u] = x\,u_x + t\,u_t$ corresponds to the symmetry transformation

$$M: \; u(x,t) \to \bar{u}(x,t;\alpha) = u(e^{\alpha}x, e^{\alpha}t) \tag{4.20}$$

expressing the invariance of the wave equation under a scale change of x and t. [The reader may show that the transformations (4.16) – (4.19) also express symmetries of the wave equation.]

It is remarkable that each of the above PDEs admits an infinite set of symmetry transformations [1]. An effective method for finding such infinite sets is the use of a *recursion operator*, which produces a new symmetry characteristic every time it acts on a known characteristic. More will be said on recursion operators in Sec. 6.

5. The Lie algebra of symmetries

As is well known [1], the set of symmetries of a PDE $F[u]=0$ has the structure of a Lie algebra. Let us demonstrate this property in the context of our abstract formalism.

Proposition 5.1. Let $\mathcal{L}$ be the set of characteristic derivatives Δ_Q with respect to the symmetry characteristics $Q[u]$ of the PDE $F[u]=0$. The set $\mathcal{L}$ is a (finite- or infinite-dimensional) Lie subalgebra of the Lie algebra of characteristic derivatives acting on functions $F[u]$ in the jet space (cf. Sec. 3).

Proof. (*a*) Let $\Delta_Q \in \mathcal{L} \Leftrightarrow \Delta_Q F[u]=0$ (mod $F[u]$). If λ is a constant (real or complex, depending on the nature of the problem) then $(\lambda\Delta_Q)F[u] \equiv \lambda\Delta_Q F[u]=0$, which means that $\lambda\Delta_Q \in \mathcal{L}$. Given that $\lambda\Delta_Q = \Delta_{\lambda Q}$ [see Eq. (3.1)] we conclude that, if $Q[u]$ is a symmetry characteristic of $F[u]=0$, then so is $\lambda Q[u]$.

(*b*) Let $\Delta_1 \in \mathcal{L}$ and $\Delta_2 \in \mathcal{L}$ be characteristic derivatives with respect to the symmetry characteristics $Q_1[u]$ and $Q_2[u]$, respectively. Then, $\Delta_1 F[u]=0$, $\Delta_2 F[u]=0$, and hence, $(\Delta_1+\Delta_2)F[u] \equiv \Delta_1 F[u]+\Delta_2 F[u]=0$; therefore, $(\Delta_1+\Delta_2) \in \mathcal{L}$. It also follows from Eq. (3.2) that, if $Q_1[u]$ and $Q_2[u]$ are symmetry characteristics of $F[u]=0$, then so is their sum $Q_1[u]+Q_2[u]$.

(*c*) Let $\Delta_1 \in \mathcal{L}$ and $\Delta_2 \in \mathcal{L}$, as above. Then, by (4.15),

$$\Delta_1 F[u] = \hat{L}_1 F[u] , \quad \Delta_2 F[u] = \hat{L}_2 F[u] .$$

Now, by the definition of the Lie bracket and the linearity of both Δ_i and $\hat{L}_i$ ($i=1,2$) we have:

$$[\Delta_1 , \Delta_2] F[u] = \Delta_1(\Delta_2 F[u]) - \Delta_2(\Delta_1 F[u]) = \Delta_1(\hat{L}_2 F[u]) - \Delta_2(\hat{L}_1 F[u])$$
$$\equiv (\Delta_1 \hat{L}_2 - \Delta_2 \hat{L}_1) F[u] = 0 \ \text{ mod } \ F[u]$$

We thus conclude that $[\Delta_1 , \Delta_2] \in \mathcal{L}$. Moreover, it follows from Eq. (3.3) that, if $Q_1[u]$ and $Q_2[u]$ are symmetry characteristics of $F[u]=0$, then so is the function

$$Q_{1,2}[u] = \Delta_1 Q_2[u] - \Delta_2 Q_1[u] .$$

Assume now that the PDE $F[u]=0$ has an n-dimensional symmetry algebra $\mathcal{L}$ (which may be a finite subalgebra of an infinite-dimensional symmetry Lie algebra). Let $\{\Delta_1 , \Delta_2 , \dots , \Delta_n\} \equiv \{\Delta_k\}$, with corresponding symmetry characteristics $\{Q_k\}$, be a set of n linearly independent operators that constitute a basis of $\mathcal{L}$, and let Δ_i , Δ_j be any two elements of this basis. Given that $[\Delta_i , \Delta_j] \in \mathcal{L}$, this Lie bracket must be expressible as a linear combination of the $\{\Delta_k\}$, with constant coefficients. We write

$$[\Delta_i , \Delta_j] = \sum_{k=1}^{n} c_{ij}^k \Delta_k \tag{5.1}$$

where the coefficients of the Δ_k in the sum are the antisymmetric *structure constants* of the Lie algebra $\mathcal{L}$ in the basis $\{\Delta_k\}$.

The operator relation (5.1) can be expressed in an equivalent, characteristic form by allowing the operators on both sides to act on u and by using the fact that $\Delta_k u = Q_k[u]$:

$$[\Delta_i , \Delta_j]u = \left(\sum_{k=1}^{n} c_{ij}^k \Delta_k \right) u = \sum_{k=1}^{n} c_{ij}^k (\Delta_k u) \Rightarrow$$

$$\Delta_i Q_j[u] - \Delta_j Q_i[u] = \sum_{k=1}^{n} c_{ij}^k Q_k[u] \tag{5.2}$$

Example 5.1. One of the several forms of the *Korteweg-de Vries (KdV) equation* is

$$F[u] \equiv u_t + uu_x + u_{xxx} = 0 .$$

The symmetry condition (4.14) is written

$$S(Q;u) \equiv Q_t + Q u_x + u Q_x + Q_{xxx} = 0 \mod F[u] \tag{5.3}$$

where $S(Q;u) = \Delta_Q F[u]$. The KdV equation admits a symmetry Lie algebra of infinite dimensions [1]. This algebra has a finite, 4-dimensional subalgebra $\mathcal{L}$ of *point transformations*. A symmetry operator (characteristic derivative) Δ_Q is determined by its corresponding characteristic $Q[u] = \Delta_Q u$. Thus, a basis $\{\Delta_1, ..., \Delta_4\}$ of $\mathcal{L}$ corresponds to a set of four independent characteristics $\{Q_1, ..., Q_4\}$. Such a basis of characteristics is the following:

$$Q_1[u] = u_x , \quad Q_2[u] = u_t , \quad Q_3[u] = tu_x - 1 , \quad Q_4[u] = xu_x + 3tu_t + 2u$$

The $Q_1, ..., Q_4$ satisfy the PDE (5.3), since, as we can show,

$$S(Q_1;u) = D_x F[u] , \quad S(Q_2;u) = D_t F[u] , \quad S(Q_3;u) = tD_x F[u] ,$$
$$S(Q_4;u) = (5 + xD_x + 3tD_t) F[u]$$

[Note once more that $\Delta_Q F[u]$ is of the form (4.15) in each case.] Let us now see two examples of calculating the structure constants of $\mathcal{L}$ by application of (5.2). We have:

$$\Delta_1 Q_2 - \Delta_2 Q_1 = \Delta_1 u_t - \Delta_2 u_x = (\Delta_1 u)_t - (\Delta_2 u)_x = (Q_1)_t - (Q_2)_x = (u_x)_t - (u_t)_x = 0$$
$$\equiv \sum_{k=1}^{4} c_{12}^k Q_k$$

Since the Q_k are linearly independent, we must necessarily have $c_{12}^k = 0$, $k = 1,2,3,4$. Also,

$$\Delta_2 Q_3 - \Delta_3 Q_2 = \Delta_2 (tu_x - 1) - \Delta_3 u_t = t(\Delta_2 u)_x - (\Delta_3 u)_t = t(Q_2)_x - (Q_3)_t$$
$$= tu_{tx} - (u_x + tu_{xt}) = -u_x = -Q_1 \equiv \sum_{k=1}^{4} c_{23}^k Q_k$$

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Therefore, $c_{23}^{1}=-1$, $c_{23}^{2}=c_{23}^{3}=c_{23}^{4}=0$.

6. Recursion operators

Let $\delta u=\alpha Q[u]$ be an infinitesimal symmetry of the PDE $F[u]=0$, where $Q[u]$ is the symmetry characteristic. For any solution $u(x^k)$ of this PDE, the function $Q[u]$ satisfies the linear PDE

$$S(Q;u)\equiv \Delta_Q F[u]=0 \tag{6.1}$$

Because of the linearity of (6.1) in Q, the sum $Q_1[u]+Q_2[u]$ of two solutions of this PDE, as well as the multiple $\lambda Q[u]$ of any solution by a constant, also are solutions of (6.1) for a given u. Thus, for any solution u of $F[u]=0$, the solutions $\{Q[u]\}$ of the PDE (6.1) form a linear space, which we call S_u.

A *recursion operator* $\hat{R}$ is a linear operator that maps the space S_u into itself. Thus, if $Q[u]$ is a symmetry characteristic of $F[u]=0$ (i.e., a solution of (6.1) for a given u) then so is $\hat{R}Q[u]$:

$$S(\hat{R}Q;u)=0 \quad \text{when} \quad S(Q;u)=0 \tag{6.2}$$

Obviously, any power of a recursion operator also is a recursion operator. Thus, starting with any symmetry characteristic $Q[u]$, one may in principle obtain an infinite set of such characteristics by repeated application of the recursion operator.

A new approach to recursion operators was suggested in the early 1990s [11,15-17] (see also [8-10]) according to which a recursion operator may be viewed as an *auto-Bäcklund transformation* (BT) [18] for the symmetry condition (6.1) of the PDE $F[u]=0$. By integrating the BT, one obtains new solutions $Q'[u]$ of the linear PDE (6.1) from known ones, $Q[u]$. Typically, this type of recursion operator produces *nonlocal* symmetries in which the symmetry characteristic depends on *integrals* (rather than derivatives) of u. This approach proved to be particularly effective for matrix-valued PDEs such as the nonlinear self-dual Yang-Mills equation, of which new infinite-dimensional sets of "potential symmetries" were discovered [9,11,15].

7. An example: The chiral field equation

Let us consider the *chiral field equation*

$$F[g]\equiv (g^{-1}g_x)_x+(g^{-1}g_t)_t=0 \tag{7.1}$$

where, in general, subscripts x and t denote total derivatives D_x and D_t, respectively, and where g is a $GL(n,C)$-valued function of x and t, i.e., a complex, non-singular ($n\times n$) matrix function, differentiable for all x and t. Let $\delta g=\alpha Q[g]$ be an infinitesimal symmetry transformation for the PDE (7.1), with symmetry characteristic $Q[g]=\Delta_Q g$. It is convenient to put

$$Q[g] = g\,\Phi[g] \quad \Leftrightarrow \quad \Phi[g] = g^{-1}Q[g]\ .$$

The symmetry condition for (7.1) is

$$\Delta_Q F[g] = 0 \quad \text{mod} \quad F[g]\ .$$

This condition will yield a linear PDE for Q or, equivalently, a linear PDE for Φ. By using the properties of the characteristic derivative, we find the latter PDE to be

$$S(\Phi; g) \equiv D_x\left(\Phi_x + [g^{-1}g_x, \Phi]\right) + D_t\left(\Phi_t + [g^{-1}g_t, \Phi]\right) = 0 \ \text{ mod } \ F[g] \qquad (7.2)$$

where, as usual, square brackets denote commutators of matrices.

A useful identity that will be needed in the sequel is the following:

$$(g^{-1}g_t)_x - (g^{-1}g_x)_t + [g^{-1}g_x,\, g^{-1}g_t] = 0 \qquad (7.3)$$

Let us first consider symmetry transformations in the base space, i.e., coordinate transformations of x, t. An obvious symmetry is x-translation, $x'=x+\alpha$, given that the PDE (7.1) does not contain the independent variable x explicitly. For infinitesimal values of the parameter α, we write $\delta x=\alpha$. The symmetry characteristic is $Q[g]=g_x$, so that $\Phi[g]=g^{-1}g_x$. By substituting this expression for Φ into the symmetry condition (7.2) and by using the identity (7.3), we can verify that (7.2) is indeed satisfied:

$$S(\Phi; g) = D_x F[g] = 0 \quad \text{mod} \quad F[g]\ .$$

Similarly, for t-translation, $t'=t+\alpha$ (infinitesimally, $\delta t=\alpha$) with $Q[g]=g_t$, we find

$$S(\Phi; g) = D_t F[g] = 0 \quad \text{mod} \quad F[g]\ .$$

Another obvious symmetry of (7.1) is a scale change of both x and t: $x'=\lambda x$, $t'=\lambda t$. Setting $\lambda=1+\alpha$, where α is infinitesimal, we write $\delta x=\alpha x$, $\delta t=\alpha t$. The symmetry characteristic is $Q[g]=xg_x+tg_t$, so that $\Phi[g]=xg^{-1}g_x+tg^{-1}g_t$. Substituting for Φ into the symmetry condition (7.2) and using the identity (7.3) where necessary, we find that

$$S(\Phi; g) = \left(2 + xD_x + tD_t\right) F[g] = 0 \quad \text{mod} \quad F[g]\ .$$

Let us call $Q_1[g]=g_x$, $Q_2[g]=g_t$, $Q_3[g]=xg_x+tg_t$, and let us consider the corresponding characteristic derivative operators Δ_i defined by $\Delta_i\, g=Q_i$ (i=1,2,3). It is then straightforward to verify the following commutation relations:

$$[\Delta_1, \Delta_2]\, g = \Delta_1 Q_2 - \Delta_2 Q_1 = 0 \quad \Leftrightarrow \quad [\Delta_1, \Delta_2] = 0\ ;$$

$$[\Delta_1, \Delta_3]\, g = \Delta_1 Q_3 - \Delta_3 Q_1 = -g_x = -Q_1 = -\Delta_1 g \quad \Leftrightarrow \quad [\Delta_1, \Delta_3] = -\Delta_1\ ;$$

$$[\Delta_2\,,\Delta_3]\,g\,=\,\Delta_2 Q_3-\Delta_3 Q_2=-g_t=-Q_2=-\,\Delta_2 g\quad\Leftrightarrow\quad[\Delta_2\,,\Delta_3]=-\,\Delta_2\,.$$

Next, we consider the "internal" transformation (i.e., transformation in the fiber space) $g'=g\Lambda$, where Λ is a non-singular constant matrix. Then,

$$F[g']=\Lambda^{-1}F[g]\,\Lambda=0\mod F[g]\,,$$

which indicates that this transformation is a symmetry of (7.1). Setting $\Lambda=1+\alpha M$, where α is an infinitesimal parameter while M is a constant matrix, we write, in infinitesimal form, $\delta g=\alpha gM$. The symmetry characteristic is $Q[g]=gM$, so that $\Phi[g]=M$. Substituting for Φ into the symmetry condition (7.2), we find:

$$S(\Phi\,;g)=\big[F[g]\,,M\big]=0\mod F[g]\,.$$

Given a matrix function $g(x,t)$ satisfying the PDE (7.1), consider the following system of PDEs for two functions $\Phi[g]$ and $\Phi'[g]$:

$$\begin{aligned}\Phi'_x&=\Phi_t+[g^{-1}g_t\,,\Phi]\\-\Phi'_t&=\Phi_x+[g^{-1}g_x\,,\Phi]\end{aligned}\tag{7.4}$$

The *integrability condition* (or consistency condition) $(\Phi'_x)_t=(\Phi'_t)_x$ of this system requires that Φ satisfy the symmetry condition (7.2); i.e., $S(\Phi\,;g)=0$. Conversely, by applying the integrability condition $(\Phi_t)_x=(\Phi_x)_t$ and by using the fact that g is a solution of $F[g]=0$, one finds that Φ' must also satisfy (7.2); i.e., $S(\Phi'\,;g)=0$.

We conclude that, for any function $g(x,t)$ satisfying the PDE (7.1), the system (7.4) is an *auto-Bäcklund transformation* (BT) [18] relating solutions Φ and Φ' of the symmetry condition (7.2) of this PDE; that is, relating different symmetries of the chiral field equation. Thus, if a symmetry characteristic $Q=g\Phi$ of the PDE (7.1) is known, a new characteristic $Q'=g\Phi'$ may be found by integrating the BT (7.4); the converse is also true. Since the BT (7.4) produces new symmetries from old ones, it may be regarded as a *recursion operator* for the PDE (7.1) [8-11,15-17].

As an example, consider the internal-symmetry characteristic $Q[g]=gM$ (where M is a constant matrix) corresponding to $\Phi[g]=M$. By integrating the BT (7.4) for Φ', we get $\Phi'=[X,M]$ and thus $Q'=g[X,M]$, where X is the "potential" of the PDE (7.1), defined by the system of PDEs

$$X_x=g^{-1}g_t\ ,\quad -X_t=g^{-1}g_x\tag{7.5}$$

Note the *nonlocal* character of the BT-produced symmetry Q', due to the presence of the potential X. Indeed, as seen from (7.5), in order to find X one has to *integrate* the chiral field g with respect to the independent variables x and t. The above process can be continued indefinitely by repeated application of the recursion operator (7.4), leading to an infinite sequence of increasingly nonlocal symmetries.

Unfortunately, as the reader may check, no new information is furnished by the BT (7.4) in the case of coordinate symmetries (for example, by applying the BT for $Q=g_x$ we get the known symmetry $Q'=g_t$). A recursion operator of the form (7.4), however, does produce new nonlocal symmetries from coordinate symmetries in related problems with more than two independent variables, such as the self-dual Yang-Mills equation [8-11,15].

8. Concluding remarks

The algebraic approach to the symmetry problem of PDEs, presented in this article, is, in a sense, an extension to matrix-valued problems of the ideas contained in [1], in much the same way as [4] and [5] constitute a generalization of the Harrison-Estabrook geometrical approach [2] to matrix-valued (as well as vector-valued and Lie-algebra-valued) PDEs. The main advantage of the algebraic approach is the bypassing of the difficulty associated with the differential-operator representation of the symmetry-generating vector fields that act on matrix-valued functions in the jet space.

It should be noted, however, that the standard methods [1,4,5] are still most effective for *calculating* symmetries of PDEs. In this regard, one needs to enrich the ideas presented in this article by describing a systematic process for evaluating (not just recognizing) symmetries, in the spirit of [4,5]. This will be the subject of a subsequent article.

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