Some aspects of the electromotive force: Educational review article

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Synopsis

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Certain aspects of the concept of the electromotive force (emf) of a "circuit", as this concept was defined in recent publications, are discussed. In particular, the independence of the emf from the conductivity of the circuit is explained and the role of the applied force in motional emf is analyzed.

1. Definition and analytical expression of the emf

In recent articles [1,2] we studied the concept of the *electromotive force* (*emf*) of a "circuit" and examined the extent to which the emf represents work per unit charge for a complete tour around the circuit. This educational note contains some additional remarks regarding the emf; it may be regarded as an addendum to the aforementioned publications.

We consider a closed path C (or *loop*) in a region of space where an electromagnetic (e/m) field exists (Fig. 1). Generally speaking, this loop will be called a *"circuit"* if a charge flow can be sustained on it. We *arbitrarily* assign a positive direction of traversing the loop C and we consider an element $\vec{dl}$ of C oriented in the positive direction.

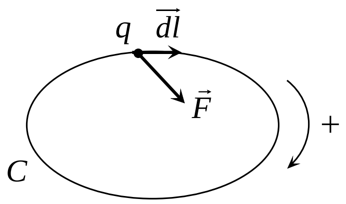

Figure 1

Let q be a *test charge*, which at time t is located at the position of $\vec{dl}$, and let $\vec{F}$ be the force on q at this time. The force $\vec{F}$ is exerted by the e/m field itself as well as, possibly, by additional *energy sources* (such as batteries or some external mechanical action) that may contribute to the generation and preservation of a current around the loop C. The *force per unit charge* at the position of $\vec{dl}$, at time t, is $\vec{f} = \vec{F}/q$. We note that $\vec{f}$ is independent of q since the e/m force on a charge is proportional to the charge.

Since, in general, neither the shape nor the size of C is required to remain fixed, and since the loop may also be in motion relative to an external observer, we will use the notation $C(t)$ to indicate the state, at time t, of a circuit of generally variable shape, size or position in space.

The *electromotive force* (*emf*) of the circuit C *at time t* is defined as the line integral of $\vec{f}$ along C, taken in the *positive* sense of C:

$$\mathcal{E}(t) = \oint_{C(t)} \vec{f}(\vec{r},t)\cdot \overrightarrow{dl} \qquad (1)$$

where $\vec{r}$ is the position vector of $\overrightarrow{dl}$ relative to the origin of our coordinate system. Obviously, the sign of the emf is dependent upon our choice of the positive direction of circulation of C. It should be noted carefully that the integral (1) is evaluated *at a given time t*. Thus, the force $\vec{f}$ must be measured *simultaneously*, at time t, at all points of C.

The force $\vec{f}$ can be attributed to two factors: (*a*) the interaction of q with the existing e/m field itself; and (*b*) the action on q by any additional energy sources that may be necessary in order to maintain a steady flow of charge on C. (This latter interaction also is *electromagnetic* in nature, even when it originates from some external mechanical action.) We write

$$\vec{f} = \vec{f}_{em} + \vec{f}_{app} \qquad (2)$$

where $\vec{f}_{em}$ is the force due to the e/m field and $\vec{f}_{app}$ is the *applied force* due to an additional energy source.

Two familiar cases of emf-driven circuits where an additional applied force is required are the following:

1. In a battery-resistor circuit [1-3] an applied force is necessary in order to carry a (conventionally *positive*) mobile charge from the negative to the positive pole of the battery, *through* the source. This force is provided by the battery itself.

2. In the case of a closed metal wire C moving in a time-independent magnetic field [2-5] the current on C is sustained for as long as the motion of C continues. This, in turn, necessitates the action of an external force on C (say, by our hand), as will be explained in Sec. 4.

Now, by (1) and (2),

$$\mathcal{E}(t) = \oint_{C(t)} \vec{f}_{em}\cdot \overrightarrow{dl} + \oint_{C(t)} \vec{f}_{app}\cdot \overrightarrow{dl} \equiv \mathcal{E}_{em}(t) + \mathcal{E}_{app}(t) \qquad (3)$$

We would like to find an analytical expression for $\mathcal{E}_{em}(t)$. So, let $\left(\vec{E}(\vec{r},t),\, \vec{B}(\vec{r},t)\right)$ be the e/m field in the region of space where the loop $C(t)$ is lying. Let q be a test charge located, at time t, at the position of $\overrightarrow{dl}$ and let $\vec{\upsilon}_{tot}$ be the total velocity of q in space, relative to some inertial frame of reference. We write

$$\vec{\upsilon}_{tot} = \vec{\upsilon} + \vec{\upsilon}_c$$

where $\vec{\upsilon}_c$ is the velocity of q along C (i.e., in a direction parallel to $\overrightarrow{dl}$) while $\vec{\upsilon}$ is the velocity of $\overrightarrow{dl}$ itself due to a possible motion in space, or just a deformation over time, of the loop $C(t)$ as a whole. The total e/m force on q is

$$\vec{F}_{em} = q[\vec{E} + (\vec{\upsilon}_{tot} \times \vec{B})] \ ,$$

so that

$$\vec{f}_{em} = \frac{\vec{F}}{q} = \vec{E} + [(\vec{\upsilon} + \vec{\upsilon}_c) \times \vec{B}] \ .$$

Hence,

$$\mathcal{E}_{em}(t) = \oint_{C(t)} \vec{E} \cdot \overrightarrow{dl} + \oint_{C(t)} (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} + \oint_{C(t)} (\vec{\upsilon}_c \times \vec{B}) \cdot \overrightarrow{dl} \ .$$

Given that $\vec{\upsilon}_c$ is parallel to $\overrightarrow{dl}$, the last integral on the right vanishes. Thus, finally,

$$\mathcal{E}_{em}(t) = \oint_{C(t)} \vec{E}(\vec{r},t) \cdot \overrightarrow{dl} + \oint_{C(t)} [\vec{\upsilon}(\vec{r},t) \times \vec{B}(\vec{r},t)] \cdot \overrightarrow{dl} \equiv \mathcal{E}_e(t) + \mathcal{E}_m(t) \qquad (4)$$

We note that, in our definition of the emf, the force per unit charge was defined as $\vec{f} = \vec{F}/q$, assuming that a replica of a test charge q is placed at every point of the circuit and that the forces $\vec{F}$ on all test charges are measured *simultaneously* at time t. Now, in the case of a conducting loop C (say, a metal wire) it is reasonable to identify q with one of the (conventionally positive) mobile free electrons. This particular identification, although logical for practical purposes, is nevertheless not necessary, given that the force $\vec{f}$ is eventually independent of q. Thus, in general, q may just be considered as a *hypothetical* test charge that is not necessarily identified with an actual mobile charge.

2. Independence from conductivity

Let $C(t)$ be a conducting loop (say, a metal wire) inside a given e/m field. The emf of C at time t is given by (3) and (4). We note from (4) that the part $\mathcal{E}_{em}$ of the total emf is independent of the velocity $\vec{\upsilon}_c$ of q *along* C (where q may be conveniently – although not necessarily – assumed to be a mobile free electron of the conductor, conventionally considered as a *positive* charge). We may physically interpret this as follows:

The e/m field creates an emf $\mathcal{E}_{em}$ that tends to generate a charge flow on C. However, this emf does not by itself determine *how fast* the mobile charges move along C. Presumably, this will depend on physical properties of the path C that are associated with its *conductivity*. (For example, in a battery-resistance circuit the potential difference at the ends of the resistance – thus the value of the electric field inside the conductor – does not by itself determine the velocity $\vec{\upsilon}_c$ of the mobile charges along the circuit, since this velocity is related to the current generated by the source, which current depends, in turn, on the resistance of the circuit, according to Ohm's law.)

Now, the role of the part $\mathcal{E}_{app}$ of the total emf (3) is to *maintain* the charge flow on $C(t)$ that is generated by $\mathcal{E}_{em}$. We thus anticipate that $\mathcal{E}_{app}$ will also be independent of $\vec{\upsilon}_c$ (this is, e.g., the case in our previous example, where $\mathcal{E}_{app}$ is equal to the voltage of the battery [1-3]). In conclusion,

> *the total emf* $\mathcal{E}(t)$ *of a conducting loop* $C(t)$ *is not dependent upon the velocity of motion of the mobile charges q along the loop.*

This leads us to a further conclusion:

> *The total emf* $\mathcal{E}(t)$ *of a conducting loop* $C(t)$ *inside an e/m field is not dependent upon the conductivity of the loop.*

This can be justified by noting that, by its definition, the force (2) does not include contributions from *resistive forces* that oppose a charge flow on C; it only contains e/m interactions that may contribute to the generation and preservation of a current in the circuit. Note, however, that the *current* itself *does* depend on the *conductivity* σ of C, according to Ohm's law ($\vec{J}=\sigma\vec{f}$) [3].

Alternatively, as argued above, the emf does not depend on $\vec{\upsilon}_c$. Now, in a steady-state situation under given electrodynamic conditions (thus, for a given $\vec{f}$) this velocity is a linear function of the *mobility* μ of q, according to the empirical relation $\vec{\upsilon}_c=\mu\vec{f}$ (by which Ohm's law is deduced). On the other hand, the conductivity of C is given by $\sigma=qn\mu$. The *density* n of mobile charges, as well as the value of q, cannot affect the value of the emf since that quantity is defined per unit charge. We thus conclude that the emf of C cannot depend on μ, as well as on n and q; hence, $\mathcal{E}$ is independent of σ.

3. Emf and the Faraday-Henry law

Consider a region of space in which a (generally time-dependent) e/m field $(\vec{E},\vec{B})$ exists. Let C be a *fixed* conducting loop in this region. There is no additional applied force on C, so (3) reduces to $\mathcal{E}(t)=\mathcal{E}_{em}(t)$. Furthermore, since C is stationary, $\vec{\upsilon}(\vec{r},t)$ vanishes identically and, by (4), $\mathcal{E}_m(t)=0$ and $\mathcal{E}_{em}(t)=\mathcal{E}_e(t)$. Thus, finally,

$$\mathcal{E}(t)=\oint_C \vec{E}(\vec{r},t)\cdot\overrightarrow{dl} \tag{5}$$

By Stokes' theorem,

$$\oint_C \vec{E}\cdot\overrightarrow{dl}=\int_S(\vec{\nabla}\times\vec{E})\cdot\overrightarrow{da}$$

where S is any open surface bounded by C (Fig. 2).

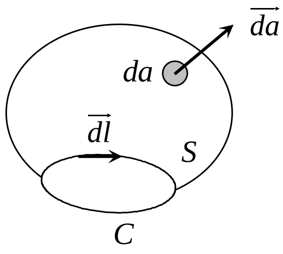

Figure 2

Moreover, by the *Faraday-Henry law*,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6}$$

So, (5) yields

$$\mathcal{E}(t) = -\frac{d}{dt} \int_S \vec{B} \cdot \overrightarrow{da} = -\frac{d}{dt} \Phi_m(t) \tag{7}$$

where

$$\Phi_m(t) = \int_S \vec{B}(\vec{r}, t) \cdot \overrightarrow{da}$$

is the *magnetic flux* through C at time t. As commented in [1], relation (7) expresses a genuine physical law, not a mere consequence of the definition of the emf.

4. Motional emf due to a static magnetic field

Let $C(t)$ be a conducting loop inside a static magnetic field $\vec{B}(\vec{r})$ (Fig. 3). The time dependence of C indicates a motion and/or a deformation of the loop over time. We will show that the emf of C at time t is given by the expression

$$\mathcal{E}(t) = \mathcal{E}_m(t) = \oint_{C(t)} [\vec{\upsilon}(\vec{r}) \times \vec{B}(\vec{r})] \cdot \overrightarrow{dl} \tag{8}$$

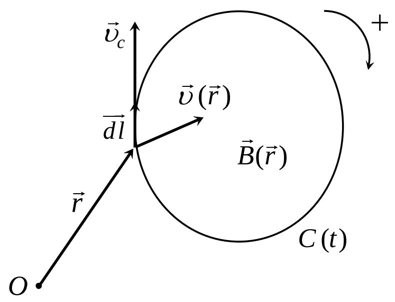

Figure 3

Let q be a mobile charge (say, a conventionally *positive* free electron) located at the position $\vec{r}$ (relative to our coordinate system) of the loop element $\overrightarrow{dl}$ at time t. As in Sec. 1, we denote the velocity of $\overrightarrow{dl}$ with respect to our frame of reference by $\vec{\upsilon}(\vec{r})$, the velocity of q along C by $\vec{\upsilon}_c$, and the total velocity of q relative to our frame by $\vec{\upsilon}_{tot} = \vec{\upsilon} + \vec{\upsilon}_c$.

Since there is no electric field in the region of interest,

$$\mathcal{E}_e(t) \equiv \oint_C \vec{E}(\vec{r},t) \cdot \overrightarrow{dl} = 0 \quad \text{and} \quad \mathcal{E}_{em}(t) = \mathcal{E}_m(t) \tag{9}$$

Also, if $\vec{f}_{app}$ is the applied force per unit charge at the position of q, at time t,

$$\mathcal{E}_{app}(t) = \oint_{C(t)} \vec{f}_{app}(\vec{r},t) \cdot \overrightarrow{dl} \tag{10}$$

The role of the applied force is to keep the current flowing. This will happen for as long as the loop C is moving or/and deforming, so that $\vec{\upsilon}(\vec{r})$ is not identically zero for all t. Why is an external force needed to keep C moving or deforming? Let us carefully analyze the situation.

The magnetic force on q is

$$\vec{F}_m = q(\vec{\upsilon}_{tot} \times \vec{B}) \quad \text{so that} \quad \vec{f}_m = \vec{\upsilon}_{tot} \times \vec{B} \ .$$

Now, imagine a temporary, local 3-dimensional rectangular system of axes (x, y, z) at the location $\vec{r}$ of q at time t. We assume, without loss of generality, that the z-axis is in the direction of $\overrightarrow{dl}$. (The orientation of the mutually perpendicular x and y-axes on the plane normal to the z-axis may be chosen arbitrarily.) Then we may write

$$\vec{f}_m = \vec{f}_{m,x} + \vec{f}_{m,y} + \vec{f}_{m,z} \equiv \vec{f}_c + \vec{f}_\perp$$

where $\vec{f}_c = \vec{f}_{m,z}$ is the component of the magnetic force *along* the loop (i.e., in a direction parallel to $\overrightarrow{dl}$) while $\vec{f}_\perp = \vec{f}_{m,x} + \vec{f}_{m,y}$ is the component *normal* to the loop (thus to $\overrightarrow{dl}$).

In a steady-state situation (steady current flow) $\vec{f}_c$ is counterbalanced by the resistive force that opposes charge motion along C (as mentioned before, this latter force does not contribute to the emf). However, to counterbalance the normal component $\vec{f}_\perp$ some external action (say, by our hand that moves or deforms the loop C) is needed in order for C to keep moving or deforming. This is precisely what the applied force $\vec{f}_{app}$ does. Clearly, this force must be *normal* to C at each point of the loop. From (10) we then conclude that

$$\mathcal{E}_{app}(t) = 0 \; .$$

Combining this with (3), (4) and (9), we finally verify the validity of (8).

It can be shown [1,3] directly from (8) that

$$\mathcal{E}(t) = -\frac{d}{dt}\,\Phi_m(t) \tag{11}$$

where $\Phi_m(t)$ is the magnetic flux through C at time t. This *looks like* (7) for a fixed geometrical loop in a time-dependent e/m field, although the origins of the two relations are different. Indeed, equation (11) is a direct consequence of the definition of the emf and may be derived from (8) essentially by mathematical manipulation (see, e.g., the Appendix in [1]). On the contrary, to derive (7) the Faraday-Henry law (6) was used. This is an *experimental* law, hence so is the expression (7) for the emf. In other words, relation (7) is not a mere mathematical consequence of the definition of the emf.

5. An example

Consider a metal bar (ab) of length h, sliding parallel to itself with constant speed υ on two parallel rails that form part of a U-shaped wire, as shown in Fig. 4. A *uniform* magnetic field $\vec{B}$, pointing into the page, fills the entire region. A circuit $C(t)$ of variable size is formed by the rectangular loop ($abcda$).

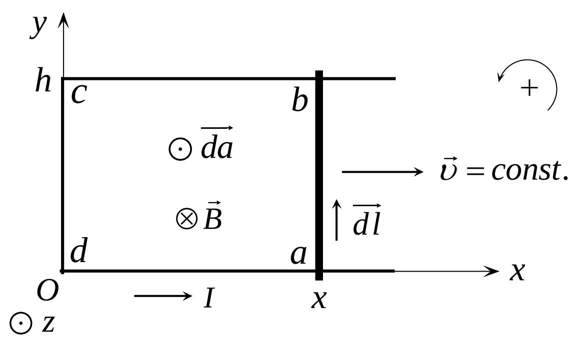

Figure 4

In Fig. 4, the *z*-axis is normal to the plane of the wire and directed toward the reader. We call $\overrightarrow{da}$ an infinitesimal normal vector representing an element of the plane surface bounded by the wire (this vector is directed toward the reader, consistently with the chosen counterclockwise direction of traversing the loop *C*). If $\hat{u}_z$ is the unit vector on the *z*-axis, then the field and the surface element are written, respectively, as $\vec{B}=-B\hat{u}_z$ (where $B=|\vec{B}|=const.$) and $\overrightarrow{da}=(da)\hat{u}_z$.

The balance of forces is shown in Fig. 5 (by $\vec{f}_r$ we denote the resistive force per unit charge, which does not contribute to the emf). Note that this diagram concerns only the *moving* part (*ab*) of the circuit, since it is in this part only that the velocity $\vec{\upsilon}$ and the applied force $\vec{f}_{app}$ are nonzero.

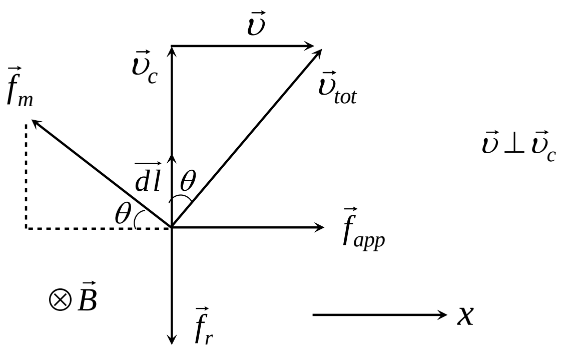

Figure 5

The emf of the circuit at time *t* is, according to (8),

$$\mathcal{E}(t) = \oint_{C(t)} (\vec{\upsilon}\times\vec{B})\cdot\overrightarrow{dl} = \int_a^b \upsilon B\, dl = \upsilon B\int_a^b dl = \upsilon Bh \ .$$

Alternatively, the magnetic flux through *C* is

$$\Phi_m(t) = \int_{S(t)} \vec{B}\cdot\overrightarrow{da} = -\int_{S(t)} B\, da = -B\int_{S(t)} da = -Bhx$$

(where *x* is the momentary position of the bar at time *t*) so that, by (11),

$$\mathcal{E}(t) = -\frac{d}{dt}\Phi_m(t) = Bh\frac{dx}{dt} = Bh\upsilon \ .$$

Now, the role of the applied force is to counterbalance the *x*-component of the magnetic force in order that the bar may move at constant speed in the *x* direction. Thus,

$$f_{app} = f_m\cos\theta = \upsilon_{tot}B\cos\theta = B\upsilon_c \ .$$

We note that, although f_{app} depends on the speed υ_c of a mobile charge along the bar, the associated part of the emf is itself independent of υ_c ! Specifically, as argued in

Sec. 4, $\mathcal{E}_{app}(t)=0$. On the other hand, in this particular example the work w of f_{app} for a complete tour around the circuit is equal to the total emf (cf. [2]): $w=\mathcal{E}=Bhv$. This equality, however, is accidental and does not reflect a more general relation between the work per unit charge and the emf. (Another such "accidental" case is the battery-resistance circuit [1-3].)

6. Summary

This article is an addendum to our study of the concept of the electromotive force (emf), as this concept was pedagogically approached in previous publications [1,2]. We have focused on some particular aspects of the subject that we felt are important enough to merit further discussion. Let us review them:

1. For a conducting loop C inside an e/m field, we explained why the emf of C does not depend on the conductivity of the loop. As "obvious" as this statement may seem, one still needs to justify it physically and to demonstrate its consistency with Ohm's law.

2. We expressed the Faraday-Henry law in terms of the emf of a closed conducting curve inside a time-dependent e/m field.

3. We studied the case of motional emf in some detail (see also [2-5]). Particularly important is the role of the applied force in this case. In addition to analyzing this role and, in the process, deriving an explicit expression for the emf, we explained why the physics of the situation is different from that of the Faraday-Henry law, despite the similar-looking forms of the emf in the two cases. Of course, as Relativity has shown, this similarity is anything but coincidental!

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