Vectors and pseudovectors in electromagnetism

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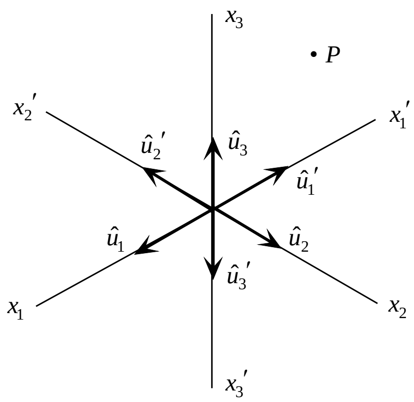
The concept of pseudovectors is simply explained. Application is made to the Maxwell equations of electromagnetism, including the case where hypothetical magnetic charges and currents are present.

1. True vectors and pseudovectors

Perhaps the simplest way to distinguish *vectors* from *pseudovectors* is to examine the way each type of object transforms under *space inversion*.



Let (x_1, x_2, x_3) be an orthogonal system of coordinates, with corresponding unit vectors $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$. This coordinate system is said to be *right-handed*, since

$$\hat{u}_1 \times \hat{u}_2 = \hat{u}_3 \ , \quad \hat{u}_2 \times \hat{u}_3 = \hat{u}_1 \ , \quad \hat{u}_3 \times \hat{u}_1 = \hat{u}_2 \qquad (1)$$

where the vector (cross) product is defined by the usual *right-hand-rule* convention.

Imagine now that we invert the directions of all three axes, thus obtaining a new coordinate system (x_1', x_2', x_3') with corresponding unit vectors

$$\hat{u}_i' = -\hat{u}_i \quad (i = 1, 2, 3) \qquad (2)$$

If we insist on using the right-hand convention, then

$$\hat{u}_1' \times \hat{u}_2' = (-\hat{u}_1) \times (-\hat{u}_2) = \hat{u}_1 \times \hat{u}_2 = \hat{u}_3 = -\hat{u}_3' \quad \text{(etc).}$$

If, however, we employ the *left-hand* convention, then

$$\hat{u}_1' \times \hat{u}_2' = \hat{u}_3' \ , \quad \hat{u}_2' \times \hat{u}_3' = \hat{u}_1' \ , \quad \hat{u}_3' \times \hat{u}_1' = \hat{u}_2' \tag{3}$$

We say that the system (x_i') is *left-handed*.[1] Thus,

- *the inversion of a right-handed coordinate system is a left-handed system.*

Let P be a point in space. The position of P does not depend, of course, on whether we choose a right-handed or a left-handed system to specify it. However, the *coordinates* of P *do* depend on this choice. Write:

$$P \equiv (x_1, x_2, x_3) \equiv (x_1', x_2', x_3') \ .$$

The two systems of coordinates are related by the set of equations

$$x_i' = -x_i \quad (i=1,2,3) \tag{4}$$

Now, consider a physical object that is described by a vector (e.g., velocity, force, electric or magnetic field, etc.). Assume that in the system (x_i) it is mathematically represented by

$$\vec{A} = A_1 \hat{u}_1 + A_2 \hat{u}_2 + A_3 \hat{u}_3 \equiv \sum_i A_i \hat{u}_i \tag{5}$$

while in the system (x_i') it is represented by

$$\vec{A}' = A_1' \hat{u}_1' + A_2' \hat{u}_2' + A_3' \hat{u}_3' \equiv \sum_i A_i' \hat{u}_i' \tag{6}$$

A *(true) vector* is a geometrical object independent of whether the coordinate system we use is right-handed or left-handed (that is, independent of the "handedness" of the underlying coordinate system). Hence,

$$\vec{A}' = \vec{A} \tag{7}$$

In view of (2), (5), (6) and (7), the components of a vector transform under space inversion according to the relations

$$A_i' = -A_i \quad (i=1,2,3) \tag{8}$$

A *pseudovector* (or *axial vector*), on the other hand, transforms differently:

$$\vec{A}' = -\vec{A} \tag{9}$$

so that, by (2), (5), (6) and (9), its components transform as follows under space inversion:

[1] Note that if only *two* axes of a right-handed coordinate system are inverted, the resulting system is still *right*-handed; if only *one* axis is inverted, the system is *left*-handed.

$$A_i{}' = A_i \quad (i=1,2,3) \tag{10}$$

Obviously, a pseudovector is *not* an invariant geometrical object since it is dependent upon the handedness of the coordinate system.

Example 1. Let each of $\vec{A}$ and $\vec{B}$ be a vector or a pseudovector. Define the vector (cross) product of these objects in the coordinate systems (x_i) and ($x_i{}'$) as follows:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \quad , \quad \vec{A}'\times\vec{B}' = \begin{vmatrix} \hat{u}_1' & \hat{u}_2' & \hat{u}_3' \\ A_1' & A_2' & A_3' \\ B_1' & B_2' & B_3' \end{vmatrix} \tag{11}$$

By taking into account relations (2) and (7) – (10), we conclude the following:

- If *both* $\vec{A}$ and $\vec{B}$ are *vectors* or *both* are *pseudovectors*, then $\vec{A}'\times\vec{B}' = -\vec{A}\times\vec{B}$ so that the cross product is a *pseudovector*.
- If *either* $\vec{A}$ *or* $\vec{B}$ is a *vector*, the other being a *pseudovector*, $\vec{A}'\times\vec{B}' = \vec{A}\times\vec{B}$ so that the cross product is a *vector*.

Example 2. Consider the *del* operator, expressed in the coordinate systems (x_i) and ($x_i{}'$) as follows:

$$\vec{\nabla} = \sum_{i=1}^{3} \hat{u}_i \frac{\partial}{\partial x_i} \quad , \quad \vec{\nabla}' = \sum_{i=1}^{3} \hat{u}_i{}' \frac{\partial}{\partial x_i{}'} \tag{12}$$

We notice that

$$\vec{\nabla}' = \sum_{i=1}^{3} (-\hat{u}_i) \frac{\partial}{\partial(-x_i)} = \vec{\nabla} \,.$$

Thus, according to (7), the del operator is a (true) *vector* operator. Then, according to Example 1,

- if $\vec{A}$ is a *vector*, its *rot* $\vec{\nabla}\times\vec{A}$ is a *pseudovector*, while
- if $\vec{B}$ is a *pseudovector*, its *rot* $\vec{\nabla}\times\vec{B}$ is a *vector*.

Definition. A quantity Φ is a (true) *scalar* if its value remains invariant under space inversion:

$$\Phi' = \Phi \tag{13}$$

A quantity Φ is a *pseudoscalar* if it changes sign under space inversion:

$$\Phi' = -\Phi \tag{14}$$

Example 3. Let each of $\vec{A}$ and $\vec{B}$ be a vector or a pseudovector. Define the scalar (dot) product of these objects in the coordinate systems (x_i) and $(x_i{}')$ as follows:

$$\vec{A}\cdot\vec{B}=\sum_{i=1}^{3}A_i B_i \ , \qquad \vec{A}'\cdot\vec{B}'=\sum_{i=1}^{3}A_i{}' B_i{}' \tag{15}$$

We observe the following:

- If *both* $\vec{A}$ and $\vec{B}$ are *vectors* or *both* are *pseudovectors*, then $\vec{A}'\cdot\vec{B}'=\vec{A}\cdot\vec{B}$ so that the dot product is a *scalar*.
- If *either* $\vec{A}$ *or* $\vec{B}$ is a *vector*, the other being a *pseudovector*, $\vec{A}'\cdot\vec{B}'=-\vec{A}\cdot\vec{B}$ so that the dot product is a *pseudoscalar*.

Example 4. Let $\vec{A}$, $\vec{B}$, $\vec{C}$ be (true) vectors. Then $\vec{B}\times\vec{C}$ is a pseudovector, so that $\vec{A}\cdot(\vec{B}\times\vec{C})$ is a pseudoscalar.

Example 5. Regarding the divergence of a vector quantity, we have the following:

- If $\vec{A}$ is a (true) *vector*, its *div* $\vec{\nabla}\cdot\vec{A}$ is a (true) *scalar*, while
- if $\vec{B}$ is a *pseudovector*, its *div* $\vec{\nabla}\cdot\vec{B}$ is a *pseudoscalar*.

Example 6. The Laplace operator

$$\nabla^2 \equiv \vec{\nabla}\cdot\vec{\nabla}=\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i{}^2} \tag{16}$$

is a scalar operator. Thus, if Φ is either a scalar or a pseudoscalar function, transforming under space inversion according to the general rule

$$\Phi'(x_1{}', x_2{}', x_3{}') = \pm\,\Phi(x_1, x_2, x_3) \tag{17}$$

(where the plus sign corresponds to a scalar while the minus sign to a pseudoscalar), then $\nabla^2\Phi$ is a scalar or a pseudoscalar function, respectively. Note also that

- *the grad* $\vec{\nabla}\Phi$ *of a scalar (pseudoscalar) function is a vector (pseudovector) function.*

2. Applications in electromagnetism

By its definition, $\vec{E} = \vec{F}_e / q$, and by the fact that the electric force $\vec{F}_e$ is a (true) vector,[2] we see that

- *the electric field is a vector.*

On the other hand, since both the magnetic force $\vec{F}_m = q(\vec{v} \times \vec{B})$ and the velocity $\vec{v}$ of a charged particle are vectors, we conclude that

- *the magnetic field is a pseudovector*

(cf. Example 1 in Sec. 1).

Consider the Maxwell equations:

$$
\begin{aligned}
&(a) \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \\
&(b) \quad \vec{\nabla} \cdot \vec{B} = 0 \\
&(c) \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
&(d) \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned}
\qquad (18)
$$

Equation (18*a*) is consistent with the fact that the electric field is a vector and the charge density ρ is a scalar function. In (18*c*) the electric field is a vector, thus its *rot* on the left-hand side is a pseudovector (cf. Example 2 in Sec. 1); this is consistent with the fact that the magnetic field is a pseudovector. In (18*d*) the magnetic field is a pseudovector, thus its *rot* on the left-hand side is a vector; this is consistent with the fact that both the electric field and the current density are vectors.

Consider the Poynting vector

$$\vec{N} = \vec{E} \times \vec{H} = \frac{1}{\mu} (\vec{E} \times \vec{B}) \ .$$

Since the electric field is a vector while the magnetic field is a pseudovector, their cross product on the right-hand side must be a vector; therefore so is the Poynting vector on the left. This was to be expected, since the direction of flow of electromagnetic energy is independent of whether our coordinate system is right-handed or left-handed.

[2] In general, a force is a physically measurable quantity that cannot depend on the handedness of our coordinate system.

3. The inclusion of magnetic charges and currents

Although magnetic charges and magnetic currents have not been observed so far in Nature, their existence cannot be precluded in principle. If such quantities are assumed to exist, the Maxwell equations must be generalized accordingly, as follows (the index *e* stands for *"electric"* while the index *m* stands for *"magnetic"*):

$$
\begin{aligned}
&(a) \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho_e}{\varepsilon_0} \\
&(b) \quad \vec{\nabla} \cdot \vec{B} = \mu_0 \, \rho_m \\
&(c) \quad \vec{\nabla} \times \vec{E} = -\mu_0 \vec{J}_m - \frac{\partial \vec{B}}{\partial t} \\
&(d) \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_e + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned}
\qquad (19)
$$

As discussed previously, $\vec{E}$ is a vector while $\vec{B}$ is a pseudovector. Moreover, the electric charge density ρ_e is a scalar function while the electric current density $\vec{J}_e$ is a vector function. Since the *div* of the magnetic field is a pseudoscalar, it follows from (19*b*) that

- *the magnetic charge density ρ_m is a pseudoscalar.*

Also, since the *rot* of the electric field is a pseudovector, it follows from (19*c*) that

- *the magnetic current density $\vec{J}_m$ is a pseudovector.*

By taking the *div* of (19*d*) and (19*c*) and by using (19*a*) and (19*b*), respectively, we find two equations of continuity:

$$\vec{\nabla} \cdot \vec{J}_e + \frac{\partial \rho_e}{\partial t} = 0 \qquad (20)$$

$$\vec{\nabla} \cdot \vec{J}_m + \frac{\partial \rho_m}{\partial t} = 0 \qquad (21)$$

The physical meaning of these relations is that the electric and the magnetic charge are separately conserved. Notice that (20) is a scalar equation while (21) is a pseudoscalar equation [the *div* of a vector (pseudovector) is a scalar (pseudoscalar)]. On the other hand, by taking the *rot* of (19*c*) and (19*d*) and by using the vector identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

together with the Maxwell system (19), we derive separate non-homogeneous wave equations for the electric and the magnetic field:

$$\nabla^2 \vec{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\varepsilon_0} \vec{\nabla} \rho_e + \mu_0 \left(\vec{\nabla} \times \vec{J}_m + \frac{\partial \vec{J}_e}{\partial t} \right) \tag{22}$$

$$\nabla^2 \vec{B} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \left(\vec{\nabla} \rho_m - \vec{\nabla} \times \vec{J}_e + \varepsilon_0 \mu_0 \frac{\partial \vec{J}_m}{\partial t} \right) \tag{23}$$

Notice that (22) is a vector equation while (23) is a pseudovector equation [recall that the *rot* of a vector (pseudovector) function is a pseudovector (vector) function].

Technically, the two wave equations (22) and (23), together with the two continuity equations (20) and (21), constitute consistency conditions for the Maxwell system (19). This system may be regarded as a sort of Bäcklund transformation relating fields and sources.[3]

[3] See https://arxiv.org/abs/1901.08058 and http://metapublishing.org/index.php/MP/catalog/book/62.