Amplitude dependence of period in one-dimensional periodic motion

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Conservation of mechanical energy furnishes a neat way to formally evaluate the period of a one-dimensional periodic motion. It is shown that the only such motion where the period does not depend on the amplitude of oscillation – thus on the total energy of the oscillating body – is simple harmonic motion.

Consider a particle of mass m, moving along the x-axis under the action of a total force $F(x)$. The position $x(t)$ of the particle as a function of time is found by integrating the second-order differential equation (Newton's second law)

$$m\,d^2x/dt^2 = F(x) \tag{1}$$

for given initial conditions $x(t_0)=x_0$ and $v(t_0)=v_0$, where $v=dx/dt$ is the velocity of the particle.

Newton's law (1) may be rewritten as a system of first-order equations:

$$dx/dt = v\,, \qquad m\,dv/dt = F(x) \tag{2}$$

Dividing these equations in order to eliminate dt, we have:

$$m\,v\,dv = F(x)\,dx = -\,dU$$

where

$$U(x) = -\int_0^x F(x')dx' \;\Leftrightarrow\; F(x) = -\,dU/dx\,.$$

Thus, $m\,v\,dv + dU = d\,(m\,v^2/2 + U) = 0 \;\Rightarrow$

$$m\,v^2/2 + U(x) \equiv T + U = E = const. \tag{3}$$

(where T = kinetic energy) which expresses conservation of total mechanical energy.

From relation (3) we get

$$(dx/dt)^2 = (2/m)\,[E-U(x)] \;\Rightarrow\; dx/dt = \pm\left\{(2/m)\,[E-U(x)]\right\}^{1/2}.$$

Integrating this first-order differential equation and taking into account the initial condition $x=x_0$ for $t=t_0$, we have:

$$\int_{x_0}^{x} \frac{\pm\,dx}{\left\{\dfrac{2}{m}\left[E-U(x)\right]\right\}^{1/2}} = t - t_0 \tag{4}$$

where the plus sign is chosen for motion in the *positive* direction ($v>0$, $x>x_0$) while the minus sign applies to motion in the *negative* direction ($v<0$, $x<x_0$). The value of the constant E in (4) may be determined by applying the initial conditions to (3): $E=mv_0^2/2+U(x_0)$, or by other physical considerations pertaining to the problem.

Let us now assume that the potential energy $U(x)$ has the form of a U-shaped potential well (Fig. 1) such that $U(0)=0$ and $U(x)>0$ for $x\neq 0$ (this arrangement is always possible because of the arbitrariness in the definition of the zero-level of the potential energy). The graph of $U(x)$ is assumed to be symmetric with respect to the axis $x=0$, which means that $U(x)$ is an *even* function: $U(-x)=U(x)$.

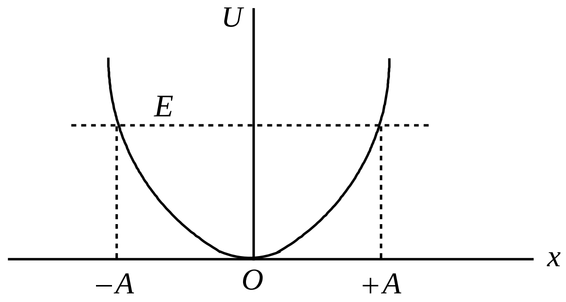

Fig. 1

Let E be the total mechanical energy of the particle. Since $E=T+U$ with $T\geq 0$, it follows that $E\geq U(x)$ for any physical motion. The motion is thus *bounded* between the points $-A$ and $+A$ of the x-axis (see Fig. 1), these points being *turning points* at which the particle stops momentarily ($E=U \Rightarrow T=0 \Rightarrow v=0$). Now, since E is constant, its value at all points equals its value at the turning points; i.e.,

$$E = U(\pm A) \tag{5}$$

The time it takes for a complete journey from $-A$ to $+A$ and back to $-A$ is found by using (4) with the appropriate sign for each direction of motion:

$$P=\int_{-A}^{A}\frac{dx}{\{\ldots\}^{1/2}}+\int_{A}^{-A}\frac{-dx}{\{\ldots\}^{1/2}} \;\Rightarrow$$

$$P=2\int_{-A}^{A}\frac{dx}{\left\{\frac{2}{m}\left[E-U(x)\right]\right\}^{1/2}} = (2m)^{1/2}\int_{-A}^{A}\left[E-U(x)\right]^{-1/2}dx \tag{6}$$

[Since $E-U(x)$ is an even function, $\int_{-A}^{A}(E-U)^{-1/2}dx = 2\int_{0}^{A}(E-U)^{-1/2}dx$.]

Given that P is fixed for a given A, the motion is periodic about the point $x=0$, with amplitude equal to A and with period P. It follows from (6) that the period P depends on A and, therefore, on the total energy E of the particle, according to (5). We will now show that an exception where P does *not* depend on A (thus on E also) is *simple harmonic motion*.

Since $U(x)$ is an even function with $U(0)=0$, it can be expanded into a Maclaurin series of the form

$$U(x) = \sum_{l=1}^{\infty} a_l \, x^{2l} \tag{7}$$

where the coefficients a_l are not necessarily all different from zero. From (5) we have

$$E = U(\pm A) = \sum_{l=1}^{\infty} a_l \, A^{2l}$$

so that

$$E - U(x) = \sum_{l=1}^{\infty} a_l \left(A^{2l} - x^{2l} \right).$$

Equation (6) then yields

$$P = \left(2m\right)^{1/2} \int_{-A}^{A} \left[\sum_{l=1}^{\infty} a_l \left(A^{2l} - x^{2l} \right) \right]^{-1/2} dx\,.$$

By setting $x/A=u \;\Leftrightarrow\; x=Au$, we get:

$$P = \left(2m\right)^{1/2} A \int_{-1}^{1} \left[\sum_{l=1}^{\infty} a_l \, A^{2l} \left(1-u^{2l} \right) \right]^{-1/2} du \tag{8}$$

It is obvious that, in general, P depends on A. The only exception where P is *not* dependent on A is the case where the following condition is satisfied: $a_l = 0$ for $l \neq 1$. That is, the only nonvanishing coefficient a_l in the series (7) is a_1. By setting $a_1 = k/2$ the potential energy (7) reduces to $U(x) = kx^2/2$, which corresponds to a restoring force of the form

$$F(x) = -\,dU/dx = -kx \tag{9}$$

The periodic motion is then *simple harmonic motion* (SHM) and the period (8) reduces to

$$P = 2(m/k)^{1/2} \int_{-1}^{1} \left(1-u^2\right)^{-1/2} du \;=\; 2(m/k)^{1/2} \left[\arcsin u\right]_{-1}^{1}$$

$$= 2(m/k)^{1/2} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] \;\Rightarrow$$

$$P = 2\pi \left(\frac{m}{k} \right)^{1/2} \equiv \frac{2\pi}{\omega} \quad \text{where} \quad \omega = \frac{2\pi}{P} = \left(\frac{k}{m} \right)^{1/2}.$$

We notice that the period of SHM is *amplitude-independent*, hence also *energy-independent*.

We may obtain the equation of motion $x=x(t)$ for SHM by using (4) with $U(x)=kx^2/2$ and $E=U(\pm A)=kA^2/2$. Let us assume first that the motion is in the positive direction, so that $x>x_0$. Setting $\omega=(k/m)^{1/2}$, we have:

$$\int_{x_0}^{x}\left(A^2-x^2\right)^{-1/2}dx=\omega\,(t-t_0)\,.$$

Using the integral formula

$$\int\left(A^2-x^2\right)^{-1/2}dx=\arcsin(x/A)+C$$

and making appropriate substitutions for constants, we find an equation of the form[1]

$$\arcsin(x/A)=\omega t+\alpha \quad\Rightarrow\quad x=A\sin(\omega t+\alpha)\,.$$

For motion in the negative direction ($x<x_0$) we choose the minus sign in (4), so that

$$\int_{x_0}^{x}\left(A^2-x^2\right)^{-1/2}dx=-\,\omega\,(t-t_0)\,.$$

This yields a result of the form[2]

$$\arcsin(x/A)=-\,\omega t+\beta \quad\Rightarrow\quad x=-A\sin(\omega t-\beta)\,.$$

Since the constant β is arbitrary (being dependent on the arbitrary constants x_0 and t_0) we may set $-\beta\equiv\pi+\alpha$, so that $x=A\sin(\omega t+\alpha)$, as before.

We conclude that the general solution of the differential equation (1) for SHM under the action of a force (9), is

$$x(t)=A\sin(\omega t+\alpha)\,.$$

Physically, A is the *amplitude* of oscillation, $\omega=(k/m)^{1/2}$ is the *angular frequency* and α is the *initial phase* (i.e., the *phase* $\omega t+\alpha$ at $t=0$).

It is of interest to examine a one-dimensional periodic motion that follows a *curved* path (where by “one-dimensional” we now mean that a single generalized coordinate – such as, e.g., an angle or a distance along the curve – is needed in order to specify the location of the particle). A nice example is that of an oscillating pendulum, shown in Fig. 2 (see also [1-3]). The position of the mass m is specified by the arc length $OA=s=l\theta$ or, equivalently, by the angle θ (in *rad*). The algebraic value of the velocity of m is $v=ds/dt=ld\theta/dt$; it may be positive or negative, depending on the direction of motion relative to the unit tangent vector $\hat{u}_T$.

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[1] Explicitly: $\alpha=\arcsin(x_0/A)-\omega t_0$.
[2] Explicitly: $\beta=\arcsin(x_0/A)+\omega t_0$.

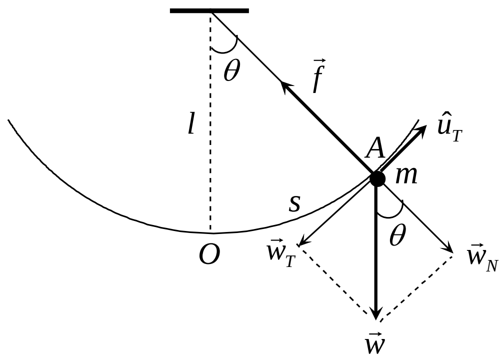

Fig. 2

The motion is governed by the tangential component $w_T = -\, mg \sin\theta$ (algebraic value) of the weight *w*. The tangential equation of motion of *m* is

$$m\, dv/dt = -mg\sin\theta \quad \Rightarrow \quad dv/dt = -g\sin\theta \tag{10}$$

We seek a conserved quantity that associates the velocity *v* with the position *θ*, in the spirit of Eq. (3). We could, of course, work with (10) directly but there is an easier way; namely, conservation of mechanical energy. This principle may be applied in view of the fact that the mass *m* is subject to the conservative force of gravity and the tension *f* of the string which, being normal to the velocity, produces no work (cf. Sec. 4.5 of [1]). The potential energy of *m* at point *A* (Fig. 2) is

$$U(\theta) = mg\,(l - l\cos\theta) = mgl\,(1-\cos\theta)\,,$$

where we have assumed that $U(0)=0$ (i.e., *U* is zero at the lowest point *O*). If *α* is the angular amplitude of oscillation (i.e., the maximum angle of deflection of the string from the vertical) then at $\theta = \pm\alpha$ the kinetic energy *T* vanishes and the total mechanical energy *E* is equal to $U(\pm\alpha)$. Applying conservation of mechanical energy between an arbitrary angle *θ* and the maximum angle $\theta=\alpha$, we have:

$$m v^2/2 + mgl\,(1-\cos\theta) = 0 + mgl\,(1-\cos\alpha) \;\;\Rightarrow\;\; \text{(after eliminating } m\text{)}$$

$$v^2 = 2gl\,(\cos\theta - \cos\alpha) \tag{11}$$

Exercise: By differentiating (11) with respect to *t* and by using the fact that $v=ld\theta/dt$, recover the equation of motion (10). Conversely, show that (11) is a direct consequence of (10). [*Hint:* Multiply (10) by *v*.]

Setting $v=ld\theta/dt$ in (11), we get a first-order differential equation:

$$d\theta/dt = \pm\left[(2g/l)(\cos\theta - \cos\alpha)\right]^{1/2},$$

which is integrated to give

$$\int_{\theta_0}^{\theta} \pm\left[\frac{2g}{l}(\cos\theta - \cos\alpha)\right]^{-1/2} d\theta = t - t_0 \;.$$

The period of oscillation is [cf. Eq. (6)]

$$\begin{aligned} P &= 2\int_{-\alpha}^{\alpha}\left[\frac{2g}{l}(\cos\theta-\cos\alpha)\right]^{-1/2} d\theta \\ &= (2l/g)^{1/2}\int_{-\alpha}^{\alpha}(\cos\theta-\cos\alpha)^{-1/2}d\theta \end{aligned} \qquad (12)$$

Obviously, P depends on the angular amplitude α. Let us assume, however, that this amplitude is very small: $\alpha <<1$. We may then make the approximations

$$\cos\theta \approx 1-\theta^2/2 \quad \text{and} \quad \cos\alpha \approx 1-\alpha^2/2 \,.$$

Furthermore, we set $\theta/\alpha=u \Leftrightarrow \theta=\alpha u$. It is then a straightforward exercise to show that (12) reduces to

$$\begin{aligned} P &= 2(l/g)^{1/2}\int_{-1}^{1}\left(1-u^2\right)^{-1/2}du \;=\; 2(l/g)^{1/2}\left[\arcsin u\right]_{-1}^{1} \\ &= 2(l/g)^{1/2}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] \;\Rightarrow \end{aligned}$$

$$P = 2\pi(l/g)^{1/2} \,,$$

which is the familiar expression for the period of oscillation of a pendulum executing simple harmonic motion for small angles of deflection from the vertical. Once again, the SHM is seen to be the only one-dimensional periodic motion in which the period does not depend on the amplitude of oscillation.

As another example, consider a body of mass m, which is moving back and forth on a U-shaped, frictionless roller-coaster track on the vertical xy-plane, where the x-axis is horizontal while the y-axis is vertical (Fig. 3). The shape of the track, which is symmetric with respect to the y-axis, is described mathematically by an equation of the form $y=f(x)$, where $f(x)$ is an *even* function and where $f(0)=0$. We want to determine the period of the oscillatory motion, given the total mechanical energy E of m (equivalently, the maximum height h reached by the body).

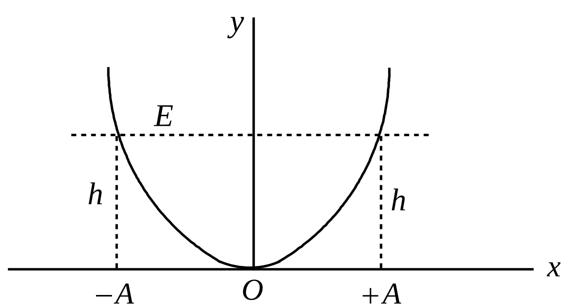

Fig. 3

Let us first take a look at the physics of the problem. The body m is sliding without friction on the roller-coaster track, moving back and forth between two extreme points at height h above the x-axis (Fig. 3). The projections of these points on this axis are $-A$ and $+A$. The body is subject to the gravitational force mg and the normal force from the track. The latter force produces no work, hence does not affect the conservation of mechanical energy (see Sec. 4.5 of [1]). The gravitational potential energy of m is $U(y)=mgy$. Along the track, where $y=f(x)$, the values of U may be expressed in terms of x:

$$U(x) = mg\,f(x) \tag{13}$$

Let E be the total mechanical energy of m. Since E is constant along the path, its value will be equal to the value of the potential energy at the extreme positions corresponding to $x=-A$ and $x=+A$ (at which positions the kinetic energy of m vanishes). That is,

$$E = U(\pm A) = mg\,f(\pm A) = mgh \tag{14}$$

The kinetic energy of the body is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

(dots indicate differentiation with respect to t) where, for $y=f(x)$,

$$\dot{y} = \frac{d}{dt}f(x) = \frac{d\,f(x)}{dx}\frac{dx}{dt} = \dot{x}f'(x) \tag{15}$$

Hence,

$$T = \frac{1}{2}m\dot{x}^2\left\{1+[f'(x)]^2\right\} \tag{16}$$

The total mechanical energy $E=T+U$ is constant along the path. By (13), (14) and (16) we have:

$$\frac{1}{2}m\dot{x}^2\left\{1+[f'(x)]^2\right\} + mg\,f(x) = mgh \tag{17}$$

The position of m on the track is specified by a single coordinate x, which plays the role of a generalized coordinate in the sense of Lagrangian dynamics. The Lagrangian function is

$$L(x,\dot{x}) = T - U = \frac{1}{2}m\dot{x}^2\left\{1+[f'(x)]^2\right\} - mg\,f(x) \tag{18}$$

The Lagrange equation for $x(t)$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{19}$$

We note that the time-derivative of *any* function of x is defined by the rule used in (15) for $f(x)$. With this in mind, it is a somewhat long but straightforward exercise to show that (18) and (19) yield the differential equation

$$\ddot{x}\left\{1+[f'(x)]^2\right\}+\dot{x}^2 f'(x)f''(x)+g f'(x)=0 \tag{20}$$

Presumably, the first-order differential equation (17) for x, expressing conservation of mechanical energy, is a *first integral* of the second-order differential equation (20). (In general, a first integral of a differential equation is a lower-order differential equation – or an algebraic relation, in the case of a first-order equation – that gives us the information that some mathematical quantity retains a constant value as a consequence of the original differential equation; see, e.g., [4].) To prove the validity of the above statement, we need to integrate (20) once with respect to t in order to derive (17). It is easier, however, to work in reverse order. We thus take the time-derivative of (17), keeping the rule (15) in mind. Not surprisingly, the result is again the differential equation (20) (show this)!

The equation of motion of m on the track is a function $x(t)$ that satisfies the differential equation (20). In principle, this second-order equation has "already" been integrated once to obtain the first-order equation (17) [which is a first integral of (20), expressing conservation of mechanical energy]. From (17) we have:

$$\dot{x}^2=\frac{2g[h-f(x)]}{1+[f'(x)]^2} \ .$$

This yields a first-order differential equation for $x(t)$:

$$\frac{dx}{dt}=\pm\left\{\frac{2g[h-f(x)]}{1+[f'(x)]^2}\right\}^{1/2}\equiv\pm\Lambda(x;h) \tag{21}$$

By assuming the initial condition $x=x_0$ for $t=t_0$, the differential equation (21) is integrated to give

$$\int_{x_0}^{x}\frac{\pm\,dx}{\Lambda(x;h)}=t-t_0 \tag{22}$$

where the plus sign is chosen for motion in the positive direction ($x>x_0$), while the minus sign applies to motion in the negative direction ($x<x_0$). This formally solves the problem of determining the position of m on the track as a function of time.

The period P of the oscillatory motion of m is the time it takes for a complete journey from the extreme position with $x=-A$ to the extreme position with $x=+A$ and back to the original position $x=-A$. To find P we use (22) with the appropriate sign for each direction of motion:

$$P=\int_{-A}^{A}\frac{dx}{\Lambda(x;h)}+\int_{A}^{-A}\frac{-dx}{\Lambda(x;h)}=2\int_{-A}^{A}\frac{dx}{\Lambda(x;h)} \ .$$

We observe that P depends on the maximum height h, thus on the total energy E of the body (notice that both the integrand *and* the limits of integration depend on h). However, P is independent of the mass of the body, as expected for a motion governed by the sole action of gravity.

References

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[3] See http://metapublishing.org/index.php/MP/catalog/book/68

[4] See https://arxiv.org/abs/1511.01788