Exponential function with any real exponent

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Problem: Let a be a positive real number. We know how to define $a^{m/n}$ with m, n integers. But, how do we define a^x for a general, real x that may be an *irrational* number, i.e., cannot be written as a quotient of integers m, n ?

Well, if it is difficult to define a function directly, we may try defining the inverse function (assuming it exists). To this end, we consider the function

$$\ln x = \int_1^x \frac{1}{t}\, dt \ , \quad x > 0 \tag{1}$$

Then,

$$(\ln x)' = 1/x$$

where the prime denotes differentiation with respect to x. Note in particular that $\ln 1 = 0$. It can also be shown [1] that, for $a, b \in R^+$, $\ln(ab) = \ln a + \ln b$, $\ln(a/b) = \ln a - \ln b$. Thus, $\ln x$ is a logarithmic function in the usual sense.

The function $\ln x$ is increasing for $x > 0$ (indeed, its derivative $1/x$ is positive for $x > 0$). Since $\ln x$ is monotone, this function is invertible. Call $\exp x$ the inverse of $\ln x$. That is,

$$y = \exp x \quad \Leftrightarrow \quad x = \ln y \ .$$

This means that

$$\exp(\ln y) = y \quad \text{and} \quad \ln(\exp x) = x \ .$$

It can be shown [1] that $\exp x$ is an exponential function in the usual sense; i.e., it has the form $\exp x = e^x$ for some $e > 0$, to be determined. We write

$$y = e^x \quad \Leftrightarrow \quad x = \ln y \quad (x \in R \ , \ y \in R^+)$$

so that

$$e^{\ln y} = y \quad \text{and} \quad \ln(e^x) = x \ .$$

Note in particular that, for $x=0$ we have $e^0 = 1$ and $\ln 1 = 0$, as required. Also, for $x=1$ we have that $\ln(e^1) = 1$ and, by the definition (1) of the logarithmic function,

$$\ln e = \int_1^e \frac{1}{t}\, dt = 1 \ .$$

We will now show that the function e^x ($x \in R$) can be expressed as the limit of a certain infinite sequence:

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$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \quad (x \in R) \tag{2}$$

Then, for any $a\in R^+$ we will have that $a=e^{\ln a} \Rightarrow$

$$a^x = e^{x\ln a} = \lim_{n\to\infty}\left(1+\frac{x\ln a}{n}\right)^n .$$

Proposition 1: Given a function $u=f(x)$ that assumes positive values for all x in its domain of definition, the derivative of $\ln[f(x)]$ is given by

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)} \tag{3}$$

Proof: $\frac{d}{dx}\ln f(x) = \frac{d(\ln u)}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{f'(x)}{f(x)}$.

Proposition 2: The derivative of e^x is given by $(e^x)' = e^x$.

Proof: $\ln(e^x) = x \Rightarrow [\ln(e^x)]' = 1 \Rightarrow (e^x)'/e^x = 1 \Rightarrow (e^x)' = e^x$, where we have used relation (3) for the derivative of $\ln(e^x)$.

Corollary: $[\exp f(x)]' = f'(x)\exp f(x)$.

Now, consider the function $g(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ $(x\in R)$. We have:

$$g'(x) = \lim_{n\to\infty}\left[n\left(1+\frac{x}{n}\right)^{n-1}\frac{1}{n}\right] = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1} = \lim_{n\to\infty}\left[\left(1+\frac{x}{n}\right)^n\left(1+\frac{x}{n}\right)^{-1}\right]$$
$$= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \cdot \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-1} = g(x)\cdot 1 = g(x).$$

Moreover, $g(0)=1$. Hence the function $y=g(x)$ satisfies the differential equation $y'=y$ with initial condition $y=1$ for $x=0$. On the other hand, the function $y=e^x$ satisfies the same differential equation with the same initial condition. Since the solution of this differential equation with given initial condition is unique, we conclude that the functions $g(x)$ and e^x must be identical. Therefore relation (2) must be true.

We note that, for $x=1$, Eq. (2) gives

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad (\simeq 2.72) \tag{4}$$

This is the formula by which the number e is usually defined.

In the same spirit we may show that another possible representation of the exponential function e^x is in the form of a power (Maclaurin) series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (x \in R) \qquad (5)$$

Indeed, notice that the x-derivative of this series is the series itself, as well as that the value of the series is equal to 1 for x=0. Although expressions (2) and (5) do not look alike, they represent the *same* function, $\exp x$! (*Note:* Two functions of x are considered identical if they have the same domain D of definition and assume equal values for all $x \in D$.)

We defined a^x (a>0, $x \in R$) in a rather indirect way by first defining the function e^x as the inverse of the function $\ln x$ and then by writing $a^x = e^{x \ln a}$. There is, however, a more direct definition of a^x. Let $x_1, x_2, \ldots, x_n, \ldots$ be *any* infinite sequence of *rational* numbers x_n such that $\lim_{n \to \infty} x_n = x \in R$. [Question: Can a sequence of rational numbers have an *irrational* limit? Yes! See, e.g., the expression (4) for e, where the latter number *is* irrational (see, e.g., [2]).] We now define a^x as follows:

$$a^x = \lim_{n \to \infty} a^{x_n} \quad (a > 0, \ x \in R).$$

Since x_n is a rational number for all n, raising a to a rational number should not be a problem! Note that the value of a^x does not depend on the specific choice of the sequence x_n, as long as the limit of this sequence is x.

References

[1] D. D. Berkey, *Calculus*, 2nd Edition (Saunders College, 1988), Chap. 8.

[2] https://mindyourdecisions.com/blog/2015/06/18/lets-prove-e-2-718-is-irrational-3-methods/