Revisiting Archimedes' principle: Buoyancy and external pressure

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A careful examination of Archimedes' principle shows that the buoyant force on a body that is either fully or partially immersed in a liquid is unaffected by the external (e.g. atmospheric) pressure, which acts both on the non-immersed part of the body (if any) and on the immersed part via Pascal's principle. The net force on the body due to the external pressure is zero and hence this pressure does not contribute to buoyancy.

1. A proper understanding of the buoyant force

In the statement of Archimedes' principle [1,2] the buoyant force on a body that is fully or partially immersed in a liquid is defined as the resultant of all elementary normal forces exerted by the liquid on the immersed surface of the body. It is then stated that this upward force is equal in magnitude to the weight of the fluid displaced by the immersed part of the body. This definition of the buoyant force turns out to be consistent with Archimedes' principle for a body that is *fully* immersed. The situation is subtler, however, in the case of a *partially* immersed floating body.

The force exerted by the fluid on the immersed surface S of the body is due to the total pressure P at the various points of S. According to Pascal's principle, this pressure is equal to the sum $P=P_l+P_0$ of the hydrostatic pressure P_l due to the liquid itself and the *constant* external pressure P_0. The buoyant force on the body is typically defined as the total force on S due to P.

In the case of a fully immersed body the immersed surface S, as well as the surface of the displaced fluid, coincides with the total surface of the body. Moreover, as will be shown, the external pressure P_0 does not contribute to the total force on any closed surface. The buoyant force is thus exclusively due to the hydrostatic pressure P_l of the liquid itself, regardless of the value of the external pressure. By the equilibrium condition for the displaced fluid, the weight of the latter is equal in magnitude and opposite in direction relative to the buoyant force (see Appendix).

In the case of a partially immersed floating body the immersed surface S is only a part of the total surface of the body. Likewise, the surface S constitutes only a part of the total surface of the displaced liquid. The non-immersed surface of the body, as well as the top surface of the displaced liquid, is subject only to the external pressure P_0 . Now, what is typically called "buoyant force" in this case is the total force on the immersed surface S of the body, which force is due to the total pressure $P=P_l+P_0$ at each point of S. By the equilibrium condition this force is assumed to be equal in magnitude to the weight of the body. But such a "balance" of forces makes no sense, given that the weight is a fixed force while the force on the immersed surface S may vary arbitrarily by changing the external pressure P_0 (this pressure is transferred to all points of S and adds to P_l in accordance with Pascal's principle). To restore the balance of forces we must include the *downward* force on the *non-immersed* surface of the body due to the external pressure. As it turns out, this force exactly matches the *upward* Pascal-oriented force on the immersed surface S due to P_0 alone, so that, eventually, the force exerted over the *entire* surface of the body (both immersed and non-immersed) by the external pressure is zero. All we are left with, therefore, is the hydrostatic force on *S* due to the pressure P_l of the liquid alone. It is *this* force that will properly balance the weight of the body. Also, it is *this* force that will balance the weight of the displaced liquid. It is thus clear that, for Archimedes' principle to be satisfied, it is the force due to P_l (*not* the force due to the total pressure *P*) that must be identified as the buoyant force in the case of a *floating* body. For a *fully immersed* body, where the immersed surface *S* is the total surface of the body, the total force due to *P* reduces to that due to P_l ; it is thus permissible to define the former force as the buoyant force in this case.

In conclusion: For consistency with Archimedes' principle regardless of whether a body is fully or partially immersed in a liquid, we must generally define the buoyant force as the total force on the immersed surface *S* of the body due to the pressure P_l exerted *by the liquid alone*. Moreover, as shown below, the (constant) external pressure P_0 contributes no additional net force on the body as a whole.

By properly defining the buoyant force, the balance of forces for a floating body, expressed by the equilibrium condition "buoyant force = total weight of the body", determines the percentage of the total volume of the body that is immersed in the liquid (cf. Sec. 8.9 of [1]). Since, as said above, the total force on the body is independent of the external pressure, it follows that we cannot make a floating body immerse further by increasing this pressure!

2. Constant external pressure on a closed surface

We propose to show that a *constant* external pressure P_0 does not affect the total force on a body that is either fully or partially immersed in a liquid. [This pressure is felt directly on the non-immersed part (if any) as well as on the immersed part via Pascal's principle.] This means that an additional constant pressure over the *entire* surface of the body does not change the total force that would be exerted on the body by the liquid alone (i.e., if the external pressure P_0 did not exist). The force on the entire surface of the body due to a constant external pressure P_0 must thus be zero.

Proposition: Consider a closed surface *S* inside a scalar field of constant value P_0 (Fig. 1). At each infinitesimal element *ds* of *S* the field exerts a force $d\vec{F}$ normal to *ds* and having magnitude *dF* proportional to the area of this surface element (which area will also be called *ds*): $dF=P_0\, ds$. We assume that, at each point of *S*, the elementary normal force $d\vec{F}$ on the local surface element *ds* is directed *toward* the surface, i.e., opposite to the local unit vector $\hat{n}$ that is normal to *S* and directed *outward*. Then, the total force exerted on *S* by the field P_0 is zero.

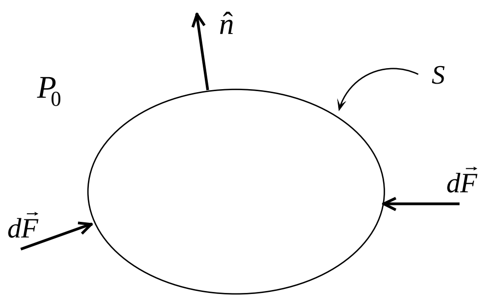


Fig. 1

Proof: We have that $d\vec{F} = -dF\,\hat{n} = -P_0\, ds\, \hat{n}$, so that the total force on S is

$$\vec{F} = \oint_S d\vec{F} = -P_0 \oint_S \hat{n}\, ds \tag{1}$$

We show that, for any closed surface S, the following integral relation is true:

$$\vec{I} \equiv \oint_S \hat{n}\, ds = 0 \tag{2}$$

It suffices to show that this vector relation is true when projected to *any* arbitrary direction. Let $\hat{b}$ be a unit vector defining such a direction. We write

$$I_b = \vec{I} \cdot \hat{b} = \oint_S \hat{b} \cdot \hat{n}\, ds\ .$$

Now, consider the constant vector field $\vec{f}(\vec{r}) = \hat{b}$. By using Gauss' integral theorem [3] we have:

$$I_b = \oint_S \vec{f}(\vec{r}) \cdot \hat{n}\, ds = \int_V (\vec{\nabla} \cdot \vec{f})\, dV = 0 \quad [\text{since } \vec{\nabla} \cdot \vec{f}(\vec{r}) = \vec{\nabla} \cdot \hat{b} = 0\,]$$

where V is the volume enclosed by the surface S. Thus, the projection of the vector-valued integral $\vec{I} \equiv \oint_S \hat{n}\, ds$ to *any* arbitrary direction vanishes, which means that the vector relation (2) is true. Accordingly, the total force $\vec{F}$ on S, given by Eq. (1), is zero.

An alternative, more "intuitive" proof of the above Proposition is the following: Since S is a closed surface, for any unit vector $\hat{n}$ normal to S at some point of this surface there exists another point of S at which the normal unit vector is directed opposite to $\hat{n}$ (of course, both unit vectors are directed *outward* relative to the surface). This is easier to understand if instead of a closed surface we consider a closed plane curve C (see Fig. 2). If we make a full trip on C, the normal unit vector $\hat{n}$ will assume all possible directions until it finally returns to its original direction at the starting point of the trip. One of these (infinitely many) directions will be the opposite of the initial direction of $\hat{n}$.

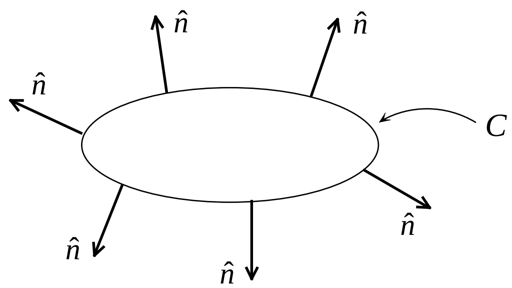


Fig. 2

Going back to our closed surface *S*, it follows from the above discussion that for every surface element $\hat{n}\,ds$ there is a corresponding element with opposite direction. This implies that $\oint_S \hat{n}\,ds = 0$ (which is an interesting mathematical result in its own right). Hence, by Eq. (1), the total force $\vec{F}$ on *S* is zero. As seen in Fig. 3, for every elementary force $d\vec{F}$ on *S* there is always an opposite force $-d\vec{F}$ acting at some other point of the surface, so that, eventually, the net force on *S* by the constant field P_0 is zero.

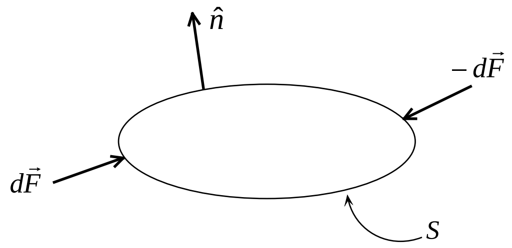
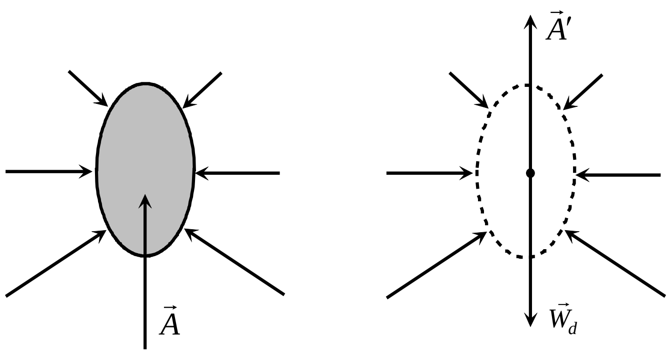


Fig. 3

In conclusion: A constant external pressure P_0 has no effect on the total force experienced by a body that is either totally or partially immersed in a liquid. In particular, the equilibrium situation of a floating body will not be altered if we increase or decrease the external pressure.

Appendix: Proof of Archimedes' principle for a fully immersed body

For a fully immersed body the principle is proven theoretically as follows: Let us call V_d and $\vec{W}_d$ the volume and the weight, respectively, of the fluid displaced by the body. Since the body is fully immersed in the liquid, V_d equals the volume of the body.



(*a*) (*b*)

Fig. 4

Part (*a*) of Fig. 4 shows an instantaneous picture of the immersed body. The word "instantaneous" is related to the fact that, in general, the body is *not* in a state of equilibrium inside the liquid. The buoyant force $\vec{A}$ is typically defined as the resultant of all elementary forces acting normally on the surface of the body by the liquid.

In part (*b*) of Fig. 4 the body has been removed and has been replaced by liquid of the same volume and shape. The surface of that section of the fluid is now subject to a total force $\vec{A}'$ (buoyant force) from the surrounding fluid. The weight $\vec{W}_d$ of this fluid section is equal to the weight of the fluid that had previously been displaced by the body, while the line of action of $\vec{W}_d$ passes through the center of gravity of the displaced fluid.

In contrast to the submerged body, the part of the liquid that replaced the body is in a state of equilibrium since it is a portion of a fluid at rest. Hence,

$$\vec{A}' + \vec{W}_d = 0 \quad \Rightarrow \quad \vec{A}' = -\vec{W}_d \ .$$

Now, the buoyant force on the body is the same as the buoyant force on the part of the fluid replacing the body (i.e., $\vec{A} = \vec{A}'$) since the elementary forces exerted by a fluid on a surface are independent of the nature of the surface [1]. Thus, finally, the buoyant force exerted by the fluid on the body is $\vec{A} = -\vec{W}_d$. The direction of $\vec{A}$ is upward (i.e., opposite to the direction of $\vec{W}_d$) while its magnitude is $A = W_d = \rho g V_d$, where ρ is the density of the liquid.

We note that the total force on the surface of the fully immersed body contains contributions from the constant external pressure P_0, which pressure is transferred via Pascal's principle to all points of the liquid. As we have shown, however, the net force due to P_0 over any closed surface (hence the surface of the body) is zero. Thus the buoyant force $\vec{A}$, which was defined as the total force exerted by the surrounding liquid, is eventually *independent* of the external pressure P_0 and equal to the force due to the pressure P_l of the liquid itself.

The case of a *partly* immersed floating body is subtler, as we discussed earlier. Consistency with Archimedes' principle suggests that the properly defined buoyant force is the force due to the pressure P_l exerted on the immersed part of the body by the liquid alone, while the external pressure P_0 (acting on both immersed and non-immersed parts of the body) contributes no extra net force on the body as a whole. Thus the buoyant force is independent of external pressure and equal in magnitude to the weight of the displaced fluid, in accordance with Archimedes' principle.

References

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2. R. Resnick, D. Halliday, K. S. Krane, *Physics:* Volume 1, 5th Edition (Wiley, 2002).
3. M. D. Greenberg, *Advanced Engineering Mathematics,* 2nd Edition (Prentice-Hall, 1998), Chap. 16.

[1] Manuscript: https://metapublishing.org/index.php/MP/catalog/book/68